Mathematics 54W

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Midterm Solutions—March 03, 2005

1. (15 pts) Find a matrix X which satisfies the given conditions if possible. If not, explain why not.

   (a) (3 pts) $2X + \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$.

   $$\begin{aligned} 2X &= \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -2 - 2 \\ 2 - 2 \end{pmatrix} \\ X &= \begin{pmatrix} -1 - 1 \\ 1 - 1 \end{pmatrix} \end{aligned}$$

   (b) (3 pts) $X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 2 & 5 & 1 \end{pmatrix}$.
   $X = \begin{pmatrix} 5 & 3 & 1 \\ 1 & -7 & -1 \end{pmatrix}$.

   (c) (3 pts) $X = \begin{pmatrix} 3 & -2 & 0 \\ 2 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.
   This is undefined because the length of the rows of the first factor does not match the length of the columns of the second.

   (d) (2 pts) $X \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -1 \end{pmatrix}$, with X invertible.
   The elementary matrix $X := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ works here.

   (e) (2 pts) $X \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}$, with X invertible.
   The elementary matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ works.

   (f) (2 pts) $X \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}$, with X invertible.
   This is not possible since the rows spaces of the two matrices are not the same—they have different dimensions.

2. (15 pts) Let

$$A := \begin{pmatrix} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Use Gauss elimination in the standard way to:

(a) (5 pts) Find a basis for the row space of A.

(b) (5 pts) Find a basis for the column space of A from among the columns of A.

(c) (5 pts) Find a basis for the null space of A.

Solution. A is row equivalent to the matrix

$$B := \begin{pmatrix} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which is in row echelon form. Hence its nonzero rows,

$$(1 \quad -2 \quad 0 \quad 0 \quad 1 \quad 0), (0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0), (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1),$$

form a basis for $RS(A) = RS(B)$. Furthermore, the columns of A corresponding to pivot columns of B for a basis for $CS(B)$:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2, \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The free variables are x_2, x_4, x_5, and we have corresponding basis vectors for $NS(A) = NS(B)$:

$$\mathbf{w}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mathbf{w}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mathbf{w}_5 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

3. (20 pts) Let P_3 denote the vector space of polynomials p of degree at most three. You may assume that this is a vector space of dimension 4.

   (a) (5 pts) Prove that $(1, x^2, x^3 - x)$ is a linearly independent sequence in P_3.
   Suppose $a_1 1 + a_2 x^2 + a_3(x^3 - x) = 0$. Then evaluating at $x = 0$, we see that $a_1 = 0$. Evaluating at $x = 1$, we see that $a_2 = 0$. Evaluating at $x = 2$, we see that $a_3 6 = 0$, hence $a_3 = 0$ also.

   (b) (5 pts) Prove that the set W of all $p \in P_3$ such that $p(1) = p(-1)$ is a linear subspace of P_3 and that its dimension at most 3. Hint: Use the fact that $W \neq P_3$.
   Clearly the constant function 0 belongs to W, so it is not empty. If $p, q \in W$ and $a \in \mathbf{R}$, then

   $$(ap + q)(1) = ap(1) + q(1) = ap(-1) + q(-1) = (ap + q)(-1).$$

   Thus $ap + q \in W$, and W is a linear subspace. The polynomial $x \in P_3$ but is not in W, so $W \neq P_3$. Hence its dimension must be strictly smaller than 4.

   (c) (5 pts) Prove that $(1, x^2, x^3 - x)$ is an ordered basis for W.
   Since S is linearly independent, the space it spans has dimension three. Since this space is contained in W, which has dimension at most three, $span(S) = W$, so S is an ordered basis for W.

   (d) (5 pts) Find the coordinates of $(x - 1)(x + 1)$ with respect to this ordered basis.
   $(x - 1)(x + 1) = x^2 - 1 = -1 \cdot 1 + 1 \cdot x^2 + 0 \cdot (x^3 - x)$. Hence its coordinates are $(-1, 1, 0)$.

4. (10 pts) Let $A := \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) (5 pts) Find A^{-1}.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Thus $A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$.

(b) (5 pts) Write $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ as a linear combination of the columns of A.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

5. (10 pts) Let A be a 7×13 matrix.

(a) (5 pts) What is the maximum possible dimension of the column space of A? If this is achieved, what are the dimensions of the row and null spaces of A? Explain.
$CS(A) \subseteq \mathbf{R}^7$, so its maximum dimension is 7, which can be achieved since there are $13 \geq 7$ columns. The dimension of the row space will then also be 7, and that of the null space will be $13 - 7 = 6$.

(b) (5 pts) Answer the same questions for a 13×7 matrix.
In this case, the column space is a subspace of $\mathbf{R}^{13}$ and is spanned by 7 vectors, so its dimension is again at most 7. The dimension of the row space will still be 7, and the dimension of the null space will be $7 - 7 = 0$.