Math 55 First Midterm 17 Feb 2004

NAME (1 pt): ______________________________

TA (1 pt): ________________________________

Name of Neighbor to your left (1 pt): ________________________________

Name of Neighbor to your right (1 pt): ________________________________

Instructions: This is a closed book, closed notes, closed calculator, closed computer, closed network, open brain exam.

You get one point each for filling in the 4 lines at the top of this page. All other questions are worth 12 points.

Write all your answers on this exam. If you need scratch paper, ask for it, write your name on each sheet, and attach it when you turn it in (we have a stapler).

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Total	

Question 1) (12 pts) Determine whether or not the following proposition is a tautology. $a \oplus b$ means the exclusive or of a and b.

$$(q \wedge (p \oplus r)) \rightarrow (p \vee r)$$

If it is, prove it using rules for simplifying logical expressions, not using a truth table. If not, give a counterexample. Indicate what you are doing at each step (i.e. you don't need to know the Latin names of inference rules, just convince us that you know what you're doing). You may use the rule that $\neg(a \oplus b)$ is logically equivalent to either $(\neg a) \oplus b$ or $a \oplus (\neg b)$.

Answer:

$$(q \wedge (p \oplus r)) \rightarrow (p \vee r)$$

$$\Leftrightarrow \textit{(definition of} \rightarrow\textit{)}$$

$$\neg(q \wedge (p \oplus r)) \vee (p \vee r)$$

$$\Leftrightarrow \textit{(DeMorgan's Law)}$$

$$(\neg q \vee \neg(p \oplus r)) \vee (p \vee r)$$

$$\Leftrightarrow \textit{(rule that } \neg(p \oplus r) \textit{ same as } (\neg p) \oplus r\textit{)}$$

$$(\neg q \vee (\neg p \oplus r)) \vee (p \vee r)$$

$$\Leftrightarrow \textit{(definition of } \oplus\textit{)}$$

$$(\neg q \vee ((\neg p \wedge \neg r) \vee (\neg\neg p \wedge r)) \vee (p \vee r)$$

$$\Leftrightarrow \textit{(double negative)}$$

$$(\neg q \vee ((\neg p \wedge \neg r) \vee (p \wedge r)) \vee (p \vee r)$$

$$\Leftrightarrow \textit{(DeMorgan's Law)}$$

$$(\neg q \vee (\neg(p \vee r) \vee (p \wedge r)) \vee (p \vee r)$$

$$\Leftrightarrow \textit{(associativity and commutativity of } \vee\textit{)}$$

$$((p \vee r) \vee (\neg(p \vee r)) \vee (p \wedge r) \vee \neg q$$

$$\Leftrightarrow \textit{(} x \vee \neg x \textit{ is always true, where } x = p \vee r\textit{)}$$

$$(T) \vee (p \wedge r) \vee \neg q$$

$$\Leftrightarrow \textit{(} T \vee x \textit{ is always true, for any } x\textit{)}$$

$$T$$

Question 1) (12 pts) Determine whether or not the following proposition is a tautology. $c \oplus d$ means the exclusive or of c and d.

$$((t \oplus r) \wedge \neg s) \rightarrow (t \vee r)$$

If it is, prove it using rules for simplifying logical expressions, not using a truth table. If not, give a counterexample. Indicate what you are doing at each step (i.e. you don't need to know the Latin names of inference rules, just convince us that you know what you're doing). You may use the rule that $\neg(c \oplus d)$ is logically equivalent to either $(\neg c) \oplus d$ or $c \oplus (\neg d)$.

Answer:

$$(\neg s \wedge (t \oplus r)) \rightarrow (t \vee r)$$

$$\Leftrightarrow \textit{(definition of } \rightarrow\textit{)}$$

$$\neg(\neg s \wedge (t \oplus r)) \vee (t \vee r)$$

$$\Leftrightarrow \textit{(DeMorgan's Law)}$$

$$(\neg\neg s \vee \neg(t \oplus r)) \vee (t \vee r)$$

$$\Leftrightarrow \textit{(double negative)}$$

$$(s \vee \neg(t \oplus r)) \vee (t \vee r)$$

$$\Leftrightarrow \textit{(rule that } \neg(t \oplus r) \textit{ same as } (\neg t) \oplus r\textit{)}$$

$$(s \vee (\neg t \oplus r)) \vee (t \vee r)$$

$$\Leftrightarrow \textit{(definition of } \oplus\textit{)}$$

$$(s \vee ((\neg t \wedge \neg r) \vee (\neg\neg t \wedge r)) \vee (t \vee r)$$

$$\Leftrightarrow \textit{(double negative)}$$

$$(s \vee ((\neg t \wedge \neg r) \vee (t \wedge r)) \vee (t \vee r)$$

$$\Leftrightarrow \textit{(DeMorgan's Law)}$$

$$(s \vee (\neg(t \vee r) \vee (t \wedge r)) \vee (t \vee r)$$

$$\Leftrightarrow \textit{(associativity and commutativity of } \vee\textit{)}$$

$$((t \vee r) \vee (\neg(t \vee r)) \vee (t \wedge r) \vee s$$

$$\Leftrightarrow \textit{(} x \vee \neg x \textit{ is always true, where } x = t \vee r\textit{)}$$

$$(T) \vee (t \wedge r) \vee s$$

$$\Leftrightarrow \textit{(} T \vee x \textit{ is always true, for any } x\textit{)}$$

$$T$$

Question 1) (12 pts) Determine whether or not the following proposition is a tautology. $e \oplus f$ means the exclusive or of e and f.

$$((\neg a \oplus \neg c) \wedge b) \rightarrow (\neg a \vee \neg c)$$

If it is, prove it using rules for simplifying logical expressions, not using a truth table. If not, give a counterexample. Indicate what you are doing at each step (i.e. you don't need to know the Latin names of inference rules, just convince us that you know what you're doing). You may use the rule that $\neg(e \oplus f)$ is logically equivalent to either $(\neg e) \oplus f$ or $e \oplus (\neg f)$.

Answer:

$$(b \wedge (\neg a \oplus \neg c)) \rightarrow (\neg a \vee \neg c)$$

$$\Leftrightarrow \textit{(definition of} \rightarrow\textit{)}$$

$$\neg(b \wedge (\neg a \oplus \neg c)) \vee (\neg a \vee \neg c)$$

$$\Leftrightarrow \textit{(DeMorgan's Law)}$$

$$(\neg b \vee \neg(\neg a \oplus \neg c)) \vee (\neg a \vee \neg c)$$

$$\Leftrightarrow \textit{(rule that } \neg(\neg a \oplus \neg c) \textit{ same as } (\neg\neg a) \oplus \neg c\textit{)}$$

$$(\neg b \vee (\neg\neg a \oplus \neg c)) \vee (\neg a \vee \neg c)$$

$$\Leftrightarrow \textit{(double negative)}$$

$$(\neg b \vee (a \oplus \neg c)) \vee (\neg a \vee \neg c)$$

$$\Leftrightarrow \textit{(definition of } \oplus\textit{)}$$

$$(\neg b \vee ((a \wedge \neg\neg c) \vee (\neg a \wedge \neg c)) \vee (\neg a \vee \neg c)$$

$$\Leftrightarrow \textit{(double negative)}$$

$$(\neg b \vee ((a \wedge c) \vee (\neg a \wedge \neg c)) \vee (\neg a \vee \neg c)$$

$$\Leftrightarrow \textit{(DeMorgan's Law)}$$

$$(\neg b \vee (\neg(\neg a \vee \neg c) \vee (\neg a \wedge \neg c)) \vee (\neg a \vee \neg c)$$

$$\Leftrightarrow \textit{(associativity and commutativity of } \vee\textit{)}$$

$$((\neg a \vee \neg c) \vee (\neg(\neg a \vee \neg c)) \vee (\neg a \wedge \neg c) \vee \neg b$$

$$\Leftrightarrow \textit{(} x \vee \neg x \textit{ is always true, where } x = \neg a \vee \neg c\textit{)}$$

$$(T) \vee (\neg a \wedge \neg c) \vee \neg b$$

$$\Leftrightarrow \textit{(} T \vee x \textit{ is always true, for any } x\textit{)}$$

$$T$$

Question 2. (12 points) Classify each function f as one-to-one, onto, both, or neither. Justify your answers.

2.1) (4 points) $f : D \to D$, where $D = \{-3, -2, -1, 0, 1\}$, $f(x) = ((2x^3 + 1) \bmod 5) - 3$.

Answer: *f maps the 5-tuple $(-3, -2, -1, 0, 1)$ to $(-1, -3, 1, -2, 0)$. Since $f(x)$ differs for each input x and takes on each value in D, f is both one-to-one and onto.*

2.2) (4 points) Let $R_0 = \{r \mid r \text{ real and } r \geq 0\}$, and $R_1 = \{r \mid r \text{ real and } r \geq 1\}$. Let $h : R_1 \to R_0$, $h(x) = (x-1)^2$. Let $g : R_0 \to R_0$, $g(x) = 1 + \sqrt{x}$. Finally, let $f = g \circ h$.

Answer: *$f : R_1 \to R_0$ is given by $f(x) = x$. So f is one-to-one but not onto, since the range of f is R_1, a strict subset of the codomain R_0.*

2.3) (4 points) $f : H \times H \to H \times H$, where H is the set of all bit strings of length 4, and $H \times H$ is the Cartesian product, i.e. the set of all pairs of bit strings of length 4. If $b \in H$, let $b(i)$ denote the i-th bit of b, for $i = 1, 2, 3, 4$.

We interpret $b(i) = 0$ as "False" and $b(i) = 1$ as "True", so we can write logical expressions like $b(i) \vee b(i+1)$.

f is defined by $f(a, b) = (c, d)$, where a, b, c, d are all members of H, and where $c(i) = a(i) \oplus b(5-i)$ and $d(i) = a(i) \wedge \neg b(5-i)$.

Answer: *The i-th bits of c and d are determined by the i-th bit of a and $5-i$-th bit of b, so we can think of f acting on each pair of bits of a and b independently of all the other pairs of bits. A truth table shows that*

$a(i)$	$b(5-i)$	$c(i)$	$d(i)$
0	0	0	0
0	1	1	0
1	0	1	1
1	1	0	0

Since (0,0) and (1,1) both map to (0,0), f cannot be one-to-one. since the value of $(c(i), d(i))$ can never be (0,1), f cannot be onto.

Question 2. (12 points) Classify each function g as one-to-one, onto, both, or neither. Justify your answers.

2.1) (4 points) $g : E \to E$, where $E = \{-2, -1, 0, 1\}$, $g(x) = ((x^5 - 2) \bmod 4) - 2$.

Answer: *g maps the 4-tuple* $(-2, -1, 0, 1)$ *to* $(0, -1, 0, 1)$ *Since* $g(-2) = g(0)$ *, g is not one-to-one. Since* $g(x)$ *never equals -2 , g is not onto.*

2.2) (4 points) Let $T_0 = \{r \mid r \text{ real and } r \geq 0\}$, and $T_2 = \{r \mid r \text{ real and } r \geq 2\}$. Let $f : T_2 \to T_0$, $f(x) = (x-2)^2$. Let $h : T_0 \to T_2$, $h(x) = x^{1/4} + 2$. Finally, let $g = h \circ f$.

Answer: $g : T_2 \to T_2$ *is given by* $g(x) = \sqrt{x-2} + 2$*. So g is one-to-one because is it a strictly increasing function. Also g is onto, because we can solve* $g(x) = y$ *for any* $y \in T_2$ *by* $x = (y-2)^2 + 2$.

2.3) (4 points) $g : C \times C \to C \times C$, where C is the set of all bit strings of length 16, and $C \times C$ is the Cartesian product, i.e. the set of all pairs of bit strings of length 16. If $b \in C$, let $b(i)$ denote the i-th bit of b, for $i = 1, 2, ..., 16$.

We interpret $b(i) = 0$ as "False" and $b(i) = 1$ as "True", so we can write logical expressions like $b(i) \vee b(i+1)$.

g is defined by $g(p, q) = (r, s)$, where p, q, r, s are all members of H, and where $r(i) = (p(17-i) \to q(i))$ and $s(i) = (q(i) \to p(17-i))$.

Answer: *The i-th bits of r and s are determined by the i-th bit of q and* $17 - i$*-th bit of p, so we can think of g acting on each pairs of bits of p and q independently of all the other pairs of bits. A truth table shows that*

$p(17-i)$	$q(i)$	$r(i)$	$s(i)$
0	*0*	*1*	*1*
0	*1*	*1*	*0*
1	*0*	*0*	*1*
1	*1*	*1*	*1*

Since (0,0) and (1,1) both map to (1,1), g cannot be one-to-one. Since the value of $(r(i), s(i))$ *can never be (0,0), g cannot be onto.*

Question 2. (12 points) Classify each function h as one-to-one, onto, both, or neither. Justify your answers.

2.1) (4 points) $h : F \to F$, where $F = \{-2, -1, 0, 1, 2\}$, $h(x) = ((x^5 + 1) \bmod 5) - 2$.

Answer: *h maps the 5-tuple $(-2, -1, 0, 1, 2)$ to $(2, -2, -1, 0, 1)$. Since $h(x)$ differs for each input x and takes on each value in F, h is both one-to-one and onto.*

2.2) (4 points) Let $S_0 = \{r \mid r \text{ real and } r \geq 0\}$, and $S_2 = \{r \mid r \text{ real and } r \geq 2\}$. Let $g : S_2 \to S_0$, $g(x) = x^4 - 16$. Let $f : S_0 \to S_0$, $f(x) = \sqrt{x}$. Finally, let $h = f \circ g$.

Answer: *$h : S_2 \to S_0$ is given by $h(x) = \sqrt{x^4 - 16}$. So h is one-to-one since it is a strictly increasing function. Also h is onto because we can solve $h(x) = y$ for $x = (y^2 + 16)^{1/4}$, which is greater than or equal to 2 as long as $y \geq 0$.*

2.3) (4 points) $h : F \times F \to F \times F$, where F is the set of all bit strings of length 8, and $F \times F$ is the Cartesian product, i.e. the set of all pairs of bit strings of length 8. If $b \in F$, let $b(i)$ denote the i-th bit of b, for $i = 1, 2, ..., 8$.

We interpret $b(i) = 0$ as "False" and $b(i) = 1$ as "True", so we can write logical expressions like $b(i) \vee b(i + 1)$.

h is defined by $h(w, x) = (y, z)$, where w, x, y, z are all members of F, and where $y(i) = x(9 - i)$ and $z(i) = (\neg x(9 - i)) \oplus (\neg w(i))$.

Answer: *The i-th bits of y and z are determined by the i-th bit of w and $9 - i$-th bit of x, so we can think of h acting on each pair of bits of w and x independently of all the other pairs of bits. A truth table shows that*

$w(i)$	$x(9-i)$	$y(i)$	$z(i)$
0	0	0	0
0	1	1	1
1	0	0	1
1	1	1	0

Since the pair $(y(i), z(i))$ differs in every row of the truth table, h is one-to-one. Since all possible values of $(y(i), x(i))$ appear in the truth table, h is also onto.

Question 3 (12 points) In the expressions below $\log x$ means $\log_2 x$. Show your work to get full credit.

3.1) (4 points) Find the smallest integer n such that

$$f(x) = \sqrt{28x^5(\log x)^3 + 10x^4\sqrt{x}(\log x)^7 - 3x\sqrt{x^7} - 4}$$

is $O(x^{n/2})$. Note that the exponent is $n/2$.

Answer: *Of the terms being added under the square root sign,* $O(x^5(\log x)^3)$, $O(x^{4.5}(\log x)^7)$, $O(x^{4.5})$, *and* $O(1)$, *the first has the largest power of* x. *Since* $\log x = O(x^a)$ *for any* $a > 0$, *the first term is largest, and the sum is* $O(x^5(\log x)^3)$. *Taking the final square root yields* $O(x^{2.5}(\log x)^{1.5})$. *This is not* $O(x^{5/2})$ *but is* $O(n^{6/2})$ *for similar reasons, so* $n = 6$.

3.2) (4 points) Find a simple, accurate function $g(x)$ such that $f(x) = x^3 2^{3^{4^x}} - x^5 4^{3^{2^x}}$ is $O(g(x))$.

Answer: *To see which term is bigger, take logs base 2 to get* $3\log x + 3^{4^x}$ *and* $5\log x + 2\cdot 3^{2^x}$. *Each term is dominated by the exponential, so we consider just those terms, and take logs again to get* $4^x \cdot \log 3$ *and* $2^x \cdot \log 3 + 1$. *The first term is clearly largest, so overall* $f(x)$ *is* $O(x^3 2^{3^{4^x}})$,

3.3) (4 points) Find a simple, accurate function $g(x)$ such that the following function is $O(g(x))$.

$$f(x) = (3x^2 + 9x^6 - 100x^3 - 10x^2 + 27^{10^{10}})^2 \cdot (51x^3 - (\log x)^7 x^3 + \log\log\log x^x)^5 \cdot ((x^{-3}/\log x) - (x^{-4}/\log\log x))^3$$

Answer: *We consider each sum in parentheses independently. The first sum is a polynomial that is* $O(x^6)$; *after squaring we get* $O(x^{12})$. *In the second sum we simplify* $\log\log\log x^x$ *to get* $\log(\log(x\log x)) = \log(\log x + \log\log x) \le \log(2\log x) = \log 2 + \log\log x$, *which is dominated by both the other terms in the second sum, of which the second,* $(\log x)^7 x^3$, *is largest. Taking the fifth power yields* $O((\log x)^{35}x^{15})$. *Finally, factoring* $x^{-4}/\log x$ *out of the third sum yields* $(x^{-4}/\log x)\cdot(x + \log x/\log\log x)$, *from which we see that* x *is bigger than* $\log x/\log\log x$, *so we get that the third sum is* $O(x^{-3}/\log x)$. *Taking the third power yields* $O(x^{-9}/(\log x)^3)$. *Altogether the bound is the product of the bounds of the factors, namely* $O(x^{18}(\log x)^{32})$.

Question 3 (12 points) In the expressions below $\log y$ means $\log_2 y$. Show your work to get full credit.

3.1) (4 points) Find the smallest integer m such that

$$g(x) = [10^{100} - 10x^4\sqrt{x}(\log x)^7 + 39x^3(\log x)^9 + 5x(x^7)^{1/2}]^{1/3}$$

is $O(x^{m/3})$. Note that the exponent is $m/3$.

Answer: *Of the terms being added under the cube root,* $O(1)$, $O(x^{4.5}(\log x)^7)$, $O(x^3(\log x)^9)$, *and* $O(x^{4.5})$, *the second is largest. because it has the largest power of* x *and also the largest power of* $\log x$. *So the sum is* $O(x^{4.5}(\log x)^7)$. *Taking the final cube root yields* $O(x^{1.5}(\log x)^{7/3})$. *This is not* $O(x^{4/3})$ *but is* $O(n^{5/3})$ *for similar reasons, so* $m = 5$.

3.2) (4 points) Find a simple, accurate function $h(x)$ such that $g(x) = x^4 3^{2^{4^x}} - x^3 2^{4^{3^x}}$ is $O(h(x))$.

Answer: *To see which term is bigger, take logs base 2 to get* $4\log x + 2^{4^x}\log 3$ *and* $3\log x + 4^{3^x}$. *Each term is dominated by the exponential, so we consider just those terms, and take logs again to get* $4^x + \log\log 3$ *and* $3^x \cdot 2$. *The first term is clearly largest, so overall* $g(x)$ *is* $O(x^4 3^{2^{4^x}})$,

3.3) (4 points) Find a simple, accurate function $h(x)$ such that the following function is $O(h(x))$.

$$g(x) = (2x^8 - 34x^7 - 189x^9 - 3x^3 + 45^{45^{45}})^3 \cdot (15x^4 - (\log x)^8 x^3 + \log\log\log x^x)^4 \cdot ((x^{-2}\log x) - (x^{-3}/\log\log x))^2$$

Answer: *We consider each sum in parentheses independently. The first sum is a polynomial that is* $O(x^9)$; *after cubing we get* $O(x^{27})$. *In the second sum we simplify* $\log\log\log x^x$ *to get* $\log(\log(x\log x)) = \log(\log x + \log\log x) \leq \log(2\log x) = \log 2 + \log\log x$, *which is dominated by both the other terms in the second sum, of which the first,* $15x^4$, *is largest. Taking the fourth power yields* $O(x^{16})$. *Finally, factoring* $x^{-3}\log x$ *out of the third sum yields* $(x^{-3}\log x)\cdot(x + 1/(\log x\log\log x))$, *from which we see that* x *is bigger than* $1/(\log x\log\log x)$, *so we get that the third sum is* $O(x^{-2}\log x)$. *Squaring yields* $O(x^{-4}(\log x)^2)$. *Altogether the bound is the product of the bounds of the factors, namely* $O(x^{39}(\log x)^2)$.

Question 3 (12 points) In the expressions below $\log x$ means $\log_2 x$. Show your work to get full credit.

3.1) (4 points) Find the smallest integer k such that

$$f(x) = \sqrt{31x^7(\log x)^4 + 7x^6\sqrt{x}(\log x)^8 - 4x^3\sqrt{x^7} - 20x^2}$$

is $O(x^{k/2})$. Note that the exponent is $k/2$.

Answer: Of the terms being added under the square root sign, $O(x^7(\log x)^4)$, $O(x^{6.5}(\log x)^8)$, $O(x^{6.5})$, *and* $O(x^2)$, *the first has the largest power of* x, *and the sum is* $O(x^7(\log x)^4)$. *Taking the final square root yields* $O(x^{3.5}(\log x)^2)$. *This is not* $O(x^{7/2})$ *but is* $O(n^{8/2})$ *for similar reasons, so* $k = 8$.

3.2) (4 points) Find a simple, accurate function $h(x)$ such that $g(x) = x^8 3^{4^{2^x}} - x^9 4^{2^{3^x}}$ is $O(h(x))$.

Answer: To see which term is bigger, take logs base 2 to get $8\log x + 4^{2^x}\log 3$ *and* $9\log x + 2\cdot 2^{3^x}$. *Each term is dominated by the exponential, so we consider just those terms, and take logs again to get* $2^x \cdot 2 + \log\log 3$ *and* $3^x + 1$. *The second term is clearly largest, so overall* $g(x)$ *is* $O(x^9 4^{2^{3^x}})$,

3.3) (4 points) Find a simple, accurate function $g(x)$ such that the following function is $O(g(x))$.

$$h(x) = (9x^3+98x^7-234x^4-x^2+30^{40^{50}})^4\cdot(15x^2-(\log x)^6x^2+\log\log\log x^x)^2\cdot((x^{-8}/\log x)-(x^{-9}\log\log x))^4$$

Answer: We consider each sum in parentheses independently. The first sum is a polynomial that is $O(x^7)$; *after taking the fourth power we get* $O(x^{28})$. *In the second sum we simplify* $\log\log\log x^x$ *to get* $\log(\log(x\log x)) = \log(\log x + \log\log x) \le \log(2\log x) = \log 2 + \log\log x$, *which is dominated by both the other terms in the second sum, of which the second,* $(\log x)^6x^2$, *is largest. Taking the square yields* $O((\log x)^{12}x^4)$. *Finally, factoring* $x^{-9}/\log x$ *out of the third sum yields* $(x^{-9}/\log x)\cdot(x + \log x\log\log x)$, *from which we see that* x *is bigger than* $\log x\log\log x$, *so we get that the third sum is* $O(x^{-8}/\log x)$. *Taking the fourth power yields* $O(x^{-32}/(\log x)^4)$. *Altogether the bound is the product of the bounds of the factors, namely* $O((\log x)^8)$.

Question 4 (12 points) The *arithmetic mean* of two numbers x and y is defined as $A(x,y) = (x+y)/2$. The *geometric mean* of two nonnegative numbers x and y is defined as $G(x,y) = \sqrt{x \cdot y}$.

4.1) (4 points) Prove that $G(a,b) \leq A(a,b)$ for any positive a and b, and $G(a,b) < A(a,b)$ unless $a = b$.

Answer: *$G(a,b) \leq A(a,b)$ is equivalent to $\sqrt{a \cdot b} \leq (a+b)/2$, which, since both sides are positive, can be squared to get the equivalent inequality $a \cdot b \leq ((a+b)/2)^2$. Multiplying by 4 and moving everything to one side yields the equivalent inequality $a^2 - 2a \cdot b + b^2 \geq 0$, or $(a-b)^2 \geq 0$, which is true. Furthermore $(a-b)^2 > 0$ unless $a = b$.*

4.2) (8 points) Use the result of 4.1) to show that if the first 9 positive integers are placed around a circle in any order, there exist 2 integers in consecutive locations around the circle whose product is at most 24.

Answer: *The sum of the first 9 integers is 45. Therefore the sum of the arithmetic means of all 9 pairs of consecutive pairs of integers is also 45, and the mean of these arithmetic means is 45/9=5. Therefore at least one such arithmetic mean is less than or equal to 5, and the corresponding geometric mean is strictly less than 5 (since the two integers are different). Squaring, we see that the product of these two integers is strictly less than 25, i.e. at most 24.*

Question 4 (12 points) The *arithmetic mean* of two numbers a and b is defined as $AM(a,b) = (a+b)/2$. The *geometric mean* of two nonnegative numbers a and b is defined as $GM(a,b) = \sqrt{a \cdot b}$.

4.1) (4 points) Prove that $GM(x,y) \leq AM(x,y)$ for any positive x and y, and $GM(x,y) < AM(x,y)$ unless $x = y$.

Answer: $GM(x,y) \leq AM(x,y)$ *is equivalent to* $\sqrt{x \cdot y} \leq (x+y)/2$, *which, since both sides are positive, can be squared to get the equivalent inequality* $x \cdot y \leq ((x+y)/2)^2$. *Multiplying by 4 and moving everything to one side yields the equivalent inequality* $x^2 - 2x \cdot y + y^2 \geq 0$, *or* $(x-y)^2 \geq 0$, *which is true. Furthermore* $(x-y)^2 > 0$ *unless* $x = y$.

4.2) (8 points) Use the result of 4.1) to show that if the first 7 positive integers are placed around a circle in any order, there exist 2 integers in consecutive locations around the circle whose product is at most 15.

Answer: The sum of the first 7 integers is 28. Therefore the sum of the arithmetic means of all 7 pairs of consecutive pairs of integers is also 28, and the mean of these arithmetic means is 28/7=4. Therefore at least one such arithmetic mean is less than or equal to 4, and the corresponding geometric mean is strictly less than 4 (since the two integers are different). Squaring, we see that the product of these two integers is strictly less than 16, i.e. at most 15.

Question 4 (12 points) The *arithmetic mean* of two numbers r and s is defined as $am(r, s) = (r + s)/2$. The *geometric mean* of two nonnegative numbers r and s is defined as $gm(r, s) = \sqrt{r \cdot s}$.

4.1) (4 points) Prove that $gm(r, s) \leq am(r, s)$ for any positive r and s, and $gm(r, s) < am(r, s)$ unless $r = s$.

Answer: $gm(r, s) \leq am(r, s)$ *is equivalent to* $\sqrt{r \cdot s} \leq (r + s)/2$*, which, since both sides are positive, can be squared to get the equivalent inequality* $r \cdot s \leq ((r + s)/2)^2$*. Multiplying by 4 and moving everything to one side yields the equivalent inequality* $r^2 - 2r \cdot s + s^2 \geq 0$*, or* $(r - s)^2 \geq 0$*, which is true. Furthermore* $(r - s)^2 > 0$ *unless* $r = s$*.*

4.2) (8 points) Use the result of 4.1) to show that if the first 11 positive integers are placed around a circle in any order, there exist 2 integers in consecutive locations around the circle whose product is at most 35.

Answer: The sum of the first 11 integers is 66. Therefore the sum of the arithmetic means of all 11 pairs of consecutive pairs of integers is also 66, and the mean of these arithmetic means is 66/11=6. Therefore at least one such arithmetic mean is less than or equal to 6, and the corresponding geometric mean is strictly less than 6 (since the two integers are different). Squaring, we see that the product of these two integers is strictly less than 36, i.e. at most 35.

Question 5) (12 points) Determine whether the following propositions are true or false. All variable shown are integers. Show your work.

5.1) (4 points)

$$\forall n \exists m (n^2 < m)$$

Answer: This says that for any integer n we can find an integer m (that may depend on n) such that $n^2 < m$. This is true, as can be seen by choosing (for example) $m = n^2 + 1$.

5.2) (8 points)

$$\neg \forall p \exists q \forall r (p^2 - q^2 \neq 0 \text{ and } (p+q) \cdot r = 0)$$

Answer: We convert to the equivalent $\exists p \neg \exists q \forall r (p^2 - q^2 \neq 0 \text{ and } (p+q) \cdot r = 0)$, and then to the equivalent $\exists p \forall q \neg \forall r (p^2 - q^2 \neq 0 \text{ and } (p+q) \cdot r = 0)$, and then to the equivalent $\exists p \forall q \exists r \neg (p^2 - q^2 \neq 0 \text{ and } (p+q) \cdot r = 0)$, and then to the equivalent $\exists p \forall q \exists r (p^2 - q^2 = 0 \text{ or } (p+q) \cdot r \neq 0)$. We show this is true as follows. Let $p = 1$ (any nonzero value would do). Now choose any q. There are two cases to consider. If $q = -1$, then $p^2 - q^2 = 0$, and the whole clause is true for any value of r. If $q \neq -1$, then $p + q \neq 0$, and we can choose $r = 1$ (say) so that $(p+q)r \neq 0$, and again the whole clause is true.

Question 5) (12 points) Determine whether the following propositions are true or false. All variable shown are integers. Show your work.

5.1) (4 points)

$$\forall r \exists s (r^3 < s^2)$$

Answer: *This says that for any integer* r *we can find an integer* s *(that may depend on* r*) such that* $r^3 < s^2$*. This is true, as can be seen by choosing (for example)* $s = r^3 + 1$*.*

5.2) (8 points)

$$\exists a \neg \exists b \forall c (a^4 - b^4 \neq 0 \text{ and } (a - b) \cdot c = 0)$$

Answer: *We convert to the equivalent* $\exists a \forall b \neg \forall c (a^4 - b^4 \neq 0 \text{ and } (a - b) \cdot c = 0)$*, and then to the equivalent* $\exists a \forall b \exists c \neg (a^4 - b^4 \neq 0 \text{ and } (a - b) \cdot c = 0)$*, and then to the equivalent* $\exists a \forall b \exists c (a^4 - b^4 = 0 \text{ or } (a - b) \cdot c \neq 0)$*. We show this is true as follows. Let* $a = 1$ *(any nonzero value would do). Now choose any* b*. There are two cases to consider. If* $b = 1$*, then* $a^4 - b^4 = 0$*, and the whole clause is true for any value of* c*. If* $b \neq 1$*, then* $a - b \neq 0$*, and we can choose* $c = 1$ *(say) so that* $(a - b)c \neq 0$*, and again the whole clause is true.*

Question 5) (12 points) Determine whether the following propositions are true or false. All variable shown are integers. Show your work.

5.1) (4 points)

$$\forall a \exists b (a^{10} < b^4)$$

Answer: *This says that for any integer a we can find an integer b (that may depend on a) such that* $a^{10} < b^4$. *This is true, as can be seen by choosing (for example)* $b = a^{10} + 1$.

5.2) (8 points)

$$\neg \forall q \exists r \forall s (q^4 \neq r^4 \text{ and } s^2 \cdot (q + r) = 0)$$

Answer: *We convert to the equivalent* $\exists q \neg \exists r \forall s (q^4 \neq r^4 \text{ and } s^2 \cdot (q + r) = 0)$, *and then to the equivalent* $\exists q \forall r \neg \forall s (q^4 \neq r^4 \text{ and } s^2 \cdot (q + r) = 0)$, *and then to the equivalent* $\exists q \forall r \exists s \neg (q^4 \neq r^4 \text{ and } s^2 \cdot (q + r) = 0)$, *and then to the equivalent* $\exists q \forall r \exists s (q^4 = r^4 \text{ or } s^2 \cdot (q + r) \neq 0)$. *We show this is true as follows. Let* $q = 1$ *(any nonzero value would do). Now choose any* r. *There are two cases to consider. If* $r = -1$, *then* $q^4 = r^4$, *and the whole clause is true for any value of* s. *If* $r \neq -1$, *then* $q + r \neq 0$, *and we can choose* $s = 1$ *(say) so that* $s^2 \cdot (q + r) \neq 0$, *and again the whole clause is true.*