MATH 113 - S2
MID-TERM 1

1. (6 pts, 1 pt each) Answer True (T) or False (F). You do not need to write your reasoning in your answer book.

a) The associative law holds in every group.

b) The commutative law holds in every group.

c) Every cyclic group is abelian.

d) An infinite cyclic group has exactly one generator.

e) If H, K are subgroups of G, then $H \cap K$ is a subgroup of G..

f) Any two groups of order equal to four, are isomorphic.

2. (7 pts)
Show that if H, K are subgroups of an **abelian** group G, then the set

$$M = \{hk \ \ |h \in H, k \in K\}$$

is a subgroup of G

3. (7 pts)
Write down the subgroup diagram for the cyclic group $\mathbf{Z}/40\mathbf{Z}$.

Solution to mid-term 1.

1 a) T

b) F

c) T

d) F. For example, the infinite cyclic group $(\mathbb{Z}, +)$ has exactly two generators, given by "1" and "−1".

e) T. The subgroups criterion can be readily checked in this case

f) F. For example, both $G_1 = \mathbb{Z}/4\mathbb{Z}$, $G_2 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are groups of order 4, but G_1, G_2 are not isomorphic (G_1 is cyclic, while G_2 is not)

2. We apply the subgroup criterion to M:

Closure: say given

$$m_1 = h_1 k_1, \quad h_1 \in H, \ k_1 \in K$$

$$m_2 = h_2 k_2, \quad h_2 \in H, \ k_2 \in K,$$

We have

$$\begin{aligned} & m_1 \circ m_2 \\ &= (h_1 k_1) \circ (h_2 k_2) \\ &= h_1 \circ (k_1 h_2) \circ k_2 \\ &= h_1 \circ (h_2 \circ k_1) \circ k_2 \quad (\because G \text{ is abelian}) \\ &= (h_1 \circ h_2) \circ (k_1 \circ k_2) \quad \text{— } * \end{aligned}$$

Since $h_1 h_2 \in H$, $k_1 k_2 \in K$, we have, by $*$, that $m_1 \circ m_2 \in M$.

(to be cont'd)

Identity: if $e \in G$ is the identity of G, then $e \in H$, & $e \in K$, and

$$e = e \circ e$$

hence $e \in M$.

Inverse: say $m = h \circ k$, $h \in H$, $k \in K$. Then

$$m^{-1} = k^{-1} \circ h^{-1}$$
$$= h^{-1} \circ k^{-1}$$ (again because G is abelian)

Since $h^{-1} \in H$, $k^{-1} \in K$, we have $m^{-1} \in M$.

Conclusion: M is a subgroup of G.

3. The divisors of 40 are:

1, 2, 4, 5, 8, 10, 20, 40

and we get the subgroup diagram:

$$\begin{tikzcd}[column sep=small]
 & \mathbb{Z}/40\mathbb{Z} \arrow[dl, dash] \arrow[dr, dash] & \\
\langle 2 \rangle \arrow[d, dash] \arrow[drr, dash] & & \langle 5 \rangle \arrow[d, dash] \\
\langle 4 \rangle \arrow[d, dash] \arrow[drr, dash] & & \langle 10 \rangle \arrow[d, dash] \\
\langle 8 \rangle \arrow[dr, dash] & & \langle 20 \rangle \arrow[dl, dash] \\
 & \langle 40 \rangle = \langle 0 \rangle &
\end{tikzcd}$$