Math 54. Sample Answers to First Midterm

There were two versions of the midterm, distinguishable by the fact that one version had a page number on the first page, and the other one did not. The answers given here refer to the version of the test with the page number on the front page. The answer for the version without the page number is briefly given if necessary, but without details, since they are similar to the details for the answers given here.

1. (8 points) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 1 & 1 & 2 \end{bmatrix}$, if it exists. Use the algorithm introduced in Chapter 2.

The algorithm uses row reduction of the matrix $[\,A \quad I\,]$:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 6 & 7 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \\ 0 & 2 & 1 & -2 & 1 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -4 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & -11 & 3 & 6 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ 0 & 0 & -1 & -4 & 1 & 2 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -5 & 1 & 4 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ 0 & 0 & -1 & -4 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -5 & 1 & 4 \\ 0 & 1 & 0 & -3 & 1 & 1 \\ 0 & 0 & 1 & 4 & -1 & -2 \end{bmatrix}$$

Therefore the inverse is $\begin{bmatrix} -5 & 1 & 4 \\ -3 & 1 & 1 \\ 4 & -1 & -2 \end{bmatrix}$.

In the other version of the exam: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 2 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 11 & -4 & -1 \\ -14 & 5 & 2 \\ 6 & -2 & -1 \end{bmatrix}$.

2. (8 points) A matrix A and an echelon form of A are given here:

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 & -1 \\ -2 & -4 & 3 & -3 & 0 \\ 1 & 2 & -3 & 3 & 3 \\ 1 & 2 & -2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a). Write the solution set of the homogeneous system $A\vec{x} = \vec{0}$ in parametric vector form (i.e., as a linear combination of fixed vectors, in which the weights are allowed to take on arbitrary values).

Continuing the row reduction gives the following matrix in reduced echelon form:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

This gives equations $x_1 = -2x_2 + 3x_5$, $x_3 = x_4 + 2x_5$. The other variables are free, so we have the following solution in parametric vector form:

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} .$$

(b). Give a basis of $\operatorname{Nul} A$.

The above three vectors form a basis:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} .$$

(c). Give a basis of $\operatorname{Col} A$.

The pivot columns are the first and third columns, so use these columns of the *original matrix* A:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 3 \\ -3 \\ -2 \end{bmatrix} .$$

For the other version of the test: **b**. $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}$. **c**. $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$.

3. (5 points) Let A be an $m \times n$ matrix and let $\vec{b}$ be a vector in $\mathbb{R}^m$. Let $A' = \begin{bmatrix} A & \vec{b} \end{bmatrix}$ be the augmented matrix. Explain carefully why the equation $A\vec{x} = \vec{b}$ is consistent if and only if $\operatorname{rank} A = \operatorname{rank} A'$.

The system is consistent if and only if the last column of A' is not a pivot column (Theorem 2 on page 24). This happens if and only if A and A' have the same number of pivot columns (since the pivot columns of A and A' are the same, except for possibly the last column of A'). The number of pivot columns of a matrix equals its rank, so A and A' have the same number of pivot columns if and only if $\operatorname{rank} A = \operatorname{rank} A'$.

4. (8 points) Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ -1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \\ -2 \end{bmatrix}$. Let $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(a). Find a subset of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ that is a basis for H. Explain how you know it is a basis for H.

Row reduce the matrix $A = [\,\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3\,]$:

$$\begin{bmatrix} 0 & 2 & 4 \\ 1 & 0 & 1 \\ 0 & 3 & 6 \\ 0 & -1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & -1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \\ 0 & -1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the last matrix, it is easy to see that the third column equals the first column plus twice the second, which therefore also is true of the original matrix: $v_3 = v_1 + 2v_2$. So, the vectors are linearly dependent and do not give a basis.

However, $H = \text{Col}\,A$, so $\{\vec{v}_1, \vec{v}_2\}$ is a basis for H because those are the pivot columns.

Alternatively, you can use the Spanning Set Theorem in Section 4.3.

(b). Let $\mathcal{B}$ be the basis you found in part (a), and let $\vec{x} = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$. Find the $\mathcal{B}$-coordinate vector $[\vec{x}]_{\mathcal{B}}$ of $\vec{x}$.

We have $\vec{x} = \vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{v}_1 + \vec{v}_2 + (v_1 + 2v_2) = 2v_1 + 3v_2$, so

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

For the other version of the test: $\vec{v}_3 = \vec{v}_1 + 3\vec{v}_2$; $\{\vec{v}_1, \vec{v}_2\}$ is still a basis of H for the same reasons as before; and $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

5. (6 points) Let A be an $m \times n$ matrix, and let $\vec{b}$ and $\vec{c}$ be vectors in $\mathbb{R}^m$. Assume that both equations $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{c}$ are consistent. Show that the equation $A\vec{x} = \vec{b} + 7\vec{c}$ is consistent.

Since $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{c}$ are consistent, $\vec{b}$ and $\vec{c}$ lie in $\text{Col}\,A$. Since $\text{Col}\,A$ is a subspace, $7\vec{c}$ and therefore $\vec{b} + 7\vec{c}$ also lie in $\text{Col}\,A$. Thus, $A\vec{x} = \vec{b} + 7\vec{c}$ is consistent, because if $\vec{a}_1, \dots, \vec{a}_n$ are the columns of A then

$$\vec{b} + 7\vec{c} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

for some $x_1, \dots, x_n$, and then $\vec{x} = (x_1, \dots, x_n)$ is a solution of $A\vec{x} = \vec{b} + 7\vec{c}$.

For the other version of the test, change all the 7's to 6's.

6. (8 points) Use Cramer's Rule to solve for x_2 in the linear system

$$\begin{aligned} 2x_1 \qquad &+ 3x_3 = 2 \\ 3x_1 \qquad &+ 5x_3 = 3 \\ 8x_1 + x_2 \qquad &= 0 \end{aligned}$$

$$x_2 = \frac{\begin{vmatrix} 2 & 2 & 3 \\ 3 & 3 & 5 \\ 8 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 0 & 3 \\ 3 & 0 & 5 \\ 8 & 1 & 0 \end{vmatrix}} = \frac{8\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}}{-\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} = \frac{8(10-9)}{-(10-9)} = \frac{8}{-1} = -8\,.$$

(For the first step, we expanded the numerator about the bottom row and the denominator about the second column.)

For the other version of the test, $x_2 = -22$.

7. (7 points) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x_1, x_2, x_3) = (9x_1 + 4x_2 + 7x_3, 2x_1 + 3x_3)\,.$$

(a). Find the standard matrix for T.

Since $T(\vec{e}_1) = T(1,0,0) = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$, the standard matrix is

$$\begin{bmatrix} 9 & 4 & 7 \\ 2 & 0 & 3 \end{bmatrix}.$$

(b). Find a basis for the range of T. Explain your reasoning.

The range of T is a linear subspace of $\mathbb{R}^2$ containing at least the two vectors $\begin{bmatrix} 9 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$. These two vectors are linearly independent (there are two of them, and neither is a scalar multiple of the other), so by the Basis Theorem (page 179) they are a basis of $\mathbb{R}^2$.

(Or, you could notice that since the range is all of $\mathbb{R}^2$, then $\{\vec{e}_1, \vec{e}_2\}$ is a basis.)

For the other version of the test, the standard matrix is $\begin{bmatrix} 9 & 8 & 7 \\ 1 & 0 & 4 \end{bmatrix}$, and the basis of the range can be the first two columns of the matrix, or $\{\vec{e}_1, \vec{e}_2\}$, or any basis of $\mathbb{R}^2$ for that matter.