UCB Math 128A, Spring 2009: Midterm 2, Solutions

April 6, 2009

Name: ______________________________

SID: ______________________________

GSI: ______________________________

- No books, no notes, no calculators
- Justify all answers
- Do all of the 4 problems
- Exam time 50 minutes

Grading

1	/ 25
2	/ 25
3	/ 25
4	/ 25
	/100

1. (25 points)

Consider the initial-value problem with exact solution $y(t) = e^{-2t} + t$:

$$y' = 2(t - y) + 1, \quad 0 \le t \le 2, \quad y(0) = 1$$

(a) Approximate $y(2)$ using Euler's method with step size $h = 0.5$.

(b) Find the value of h necessary for an error at most 10^{-5} in $y(2)$, using the error bound below (notation as in the textbook).

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right].$$

Solution: (a)

$$\begin{aligned}
w_0 &= 1 \\
w_1 &= w_0 + hf(t_0, w_0) = 1 + 0.5 \cdot (-1) = 0.5 \\
w_2 &= w_1 + hf(t_1, w_1) = 0.5 + 0.5 \cdot 1 = 1.0 \\
w_3 &= w_2 + hf(t_2, w_2) = 1.0 + 0.5 \cdot 1 = 1.5 \\
w_4 &= w_3 + hf(t_3, w_3) = 1.5 + 0.5 \cdot 1 = 2.0 \approx y(2)
\end{aligned}$$

(b) Lipschitz constant:

$$\left|\frac{\partial f}{\partial y}\right| = 2 = L$$

Bound on second derivative of solution:

$$|y''(t)| = \left|4e^{-2t}\right| \le 4 = M$$

Find step size h to get an error of 10^{-5}:

$$10^{-5} = \frac{hM}{2L}\left[e^{L \cdot 2} - 1\right] = h\left(e^4 - 1\right) \Longrightarrow h = \frac{10^{-5}}{e^4 - 1}$$

2. (25 points)

A natural cubic spline S is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + B(x-1) - D(x-1)^3, & \text{if } 1 \le x < 2, \\ S_1(x) = 1 + b(x-2) - \frac{3}{4}(x-2)^2 + d(x-2)^3, & \text{if } 2 \le x \le 3. \end{cases}$$

If S interpolates the data $(1,1)$, $(2,1)$, and $(3,0)$, find B, D, b, and d.

Solution: Natural boundary conditions:

$$S_0''(1) = 0$$
$$S_1''(3) = -\frac{3}{2} + 6d = 0 \Longrightarrow d = \frac{1}{4}$$

Continuity of second derivative:

$$S_0''(2) = -6D$$
$$S_1''(2) = -\frac{3}{2}$$
$$S_0''(2) = S_1''(2) \Longrightarrow -6D = -\frac{3}{2} \Longrightarrow D = \frac{1}{4}$$

Interpolate the given data:

$$S_0(1) = 1$$
$$S_0(2) = 1 + B - D = 1 \Longrightarrow B = D = \frac{1}{4}$$
$$S_1(2) = 1$$
$$S_1(3) = 1 + b - \frac{3}{4} + d = b + d + \frac{1}{4} = 0 \Longrightarrow b = -\frac{1}{4} - d = -\frac{1}{2}$$

Continuity of first derivative satisfied:

$$S_0'(2) = B - 3D = -\frac{1}{2}$$
$$S_1'(2) = b = -\frac{1}{2}$$

3. (25 points)

Show how to use Newton's method to solve the equation below for x:

$$\int_{-x}^{x} e^{\sin t}\, dt = 1.$$

Use the Composite Simpson's rule with $n = 4$ intervals for the integral.

You do not have to perform any numerical calculations.

Solution: Apply Newton's method to the equation

$$f(x) = \int_{-x}^{x} e^{\sin t}\, dt - 1 = 0.$$

The integral is approximated by the Composite Simpson's rule with $h = 2x/n = x/2$:

$$\int_{-x}^{x} f(t)\, dt \approx \frac{h}{3}\left(f(-x) + 4f(-x/2) + 2f(0) + 4f(x/2) + f(x)\right).$$

The derivative is given by the fundamental theorem of calculus:

$$f'(x) = e^{\sin x} + e^{\sin -x}.$$

The Newton iteration becomes:

$$\begin{aligned}
x_0 &= \text{initial guess}, \\
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
&= x_n - \frac{\frac{x_n}{6}\left[e^{\sin -x_n} + 4e^{\sin -x_n/2} + 2e^{\sin 0} + 4e^{\sin x_n/2} + e^{\sin x_n}\right] - 1}{e^{\sin x_n} + e^{\sin -x_n}}.
\end{aligned}$$

4. (25 points)

(a) Determine the constants a, b, α such that the quadrature rule

$$\int_{-1}^{1} f(x)\,dx \approx af(-1) + af(1) + bf(\alpha) + bf(-\alpha)$$

has the highest possible degree of precision. Clearly state the degree of precision of the resulting rule.

(b) Use the quadrature rule to approximate the integral below. You can use a, b, α instead of the values calculated in (a).

$$\int_{0}^{4} e^{\sqrt{x}}\,dx.$$

Solution: (a) Match left and right hand sides for monomials $f(x) = x^n$:

$$n \text{ odd}: \begin{cases} \int_{-1}^{1} x^n\,dx = 0 \\ af(-1) + af(1) + bf(\alpha) + bf(-\alpha) = 0 \end{cases}$$

$$n \text{ even}: \begin{cases} \int_{-1}^{1} x^n\,dx = 2/(n+1) \\ af(-1) + af(1) + bf(\alpha) + bf(-\alpha) = 2a + 2b\alpha^n \end{cases}$$

Use $n = 0, 2, 4$ and solve for a, b, α:

$$\begin{cases} 2a + 2b = 2 \\ 2a + 2b\alpha^2 = \frac{2}{3} \\ 2a + 2b\alpha^4 = \frac{2}{5} \end{cases} \Longrightarrow \begin{cases} 2b(1-\alpha^2) = \frac{4}{3} \\ 2b(1-\alpha^4) = \frac{8}{5} \end{cases} \Longrightarrow \frac{1-\alpha^2}{1-\alpha^4} = \frac{5}{6}$$

$$5\alpha^4 - 6\alpha^2 + 1 = 0 \Longrightarrow \alpha^2 = \frac{3}{5} \pm \sqrt{\frac{9}{25} - \frac{1}{5}} = \frac{3}{5} \pm \frac{2}{5} = \frac{1}{5}, 1$$

The solution $\alpha^2 = 1$ is invalid, pick the positive root $\alpha = 1/\sqrt{5}$. The weights are then $a = 1/6$ and $b = 5/6$. The degree of precision of the rule is 5.

(b)

$$\int_0^4 e^{\sqrt{x}}\,dx = 2\int_{-1}^{1} e^{\sqrt{2+2t}}\,dt \approx 2\left(ae^0 + ae^{\sqrt{4}} + be^{\sqrt{2+2\alpha}} + be^{\sqrt{2-2\alpha}}\right)$$

$$= 2a(e^2+1) + 2b(e^{\sqrt{2+2\alpha}} + e^{\sqrt{2-2\alpha}})$$