Midterm Solutions—April 07, 2005

Work each problem on a separate sheet of paper. Be sure to put your name, your section number, and your GSI's name on each sheet of paper. Also, at the top of the page, in the center, write the problem number, and be sure to put the pages in order. Write clearly: explanations (with complete sentences when appropriate) will help us understand what you are doing. Note that there are problems on the back of this sheet, for a total of five problems.

1. (5 pts) Let $A := \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$, let W be the column space of A, and let $Y := \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$.

   (a) (5 pts) Use the Gram-Schmidt process to find an orthogonal basis for W.
   $w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

   (b) (5 pts) Find the orthogonal projection Y' of Y on W.

   $$Y' := \frac{(Y|w_1)}{(w_1|w_1)} + \frac{(Y|w_2)}{(w_2|w_2)} = \frac{4}{2}w_1 + \frac{-2}{2}w_2 = 2w_1 - w_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.$$

   (c) (5 pts) Find the distance from Y to W.
   This is $||Y - Y'|| = || \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} || = 1$.

   (d) (5 pts) Find X such that $A^T AX = A^T Y$. (Hint: use part (a) to save some work.)
   The equation $A^T AX = A^T Y$ says that $AX = Y'$. Let v_1 and v_2 be the columns of A. Then this equation says that $Y' = x_1 v_1 + x_2 v_2$. We know that $w_1 = v_1$ and $w_2 = 2v_2 - v_1$. Hence $Y' = 2w_1 - w_2 = 2v_1 - (2v_2 - v_1) = 3v_1 - 2v_2$. Hence $X = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

2. Let $A := \begin{pmatrix} 54 & 81 \\ -9 & 0 \end{pmatrix}$.

   (a) (2 pts) Find the characteristic polynomial of A.
   $p_A(X) = x^2 - 54X + 729$.

   (b) (3 pts) Find the eigenvalues of A.
   27 is the only eigenvalue of A.

   (c) (5 pts) Find a diagonal matrix D and a nilpotent matrix N such that $A = D + N$.
   $N = A - \begin{pmatrix} 27 & 0 \\ 0 & 27 \end{pmatrix} = \begin{pmatrix} 27 & 81 \\ -9 & -27 \end{pmatrix}$.

   (d) (5 pts) Use part (c) to find a matrix B such that that $B^3 = A$.
   We try $B = D' + N'$, where $D' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and $(N')^2 = 0$. We need $(D')^3 + 3(D')^2 N' = D + N$, so $27N' = N$. Thus $N' = \begin{pmatrix} 1 & 3 \\ -1/3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 3 \\ -1/3 & 2 \end{pmatrix}$.

3. Let $A := \begin{pmatrix} 5 & 12 \\ -2 & -5 \end{pmatrix}$.

(a) (2 pts) Find the characteristic polynomial of A.
$p_A(x) = x^2 - 1$.

(b) (3 pts) Find the eigenvalues of A.
$1, -1$

(c) (5 pts) Find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$.
$Eig_1(A) = NS\begin{pmatrix} 4 & 12 \\ -2 & -6 \end{pmatrix} = span\begin{pmatrix} 3 \\ -1 \end{pmatrix}$
$Eig_{-1}(A) = NS\begin{pmatrix} 6 & 12 \\ -2 & -4 \end{pmatrix} = span\begin{pmatrix} 2 \\ -1 \end{pmatrix}$
Hence $S = \begin{pmatrix} 3 & 2 \\ -1 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ will work. (There are other possibilities too.)

(d) (2 pts) List all possibilities for D. Explain.
D has to be a diagonal matrix with the eigenvalues of A along the diagonal, so the only other possibility is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(e) (3 pts) Find a matrix B such that $B^3 = A$.
$B = A$ will work, since $D^3 = D$.

4. Let A by an $n \times n$ matrix whose only eigenvalue is 0.

   (a) (5 pts) Is A necessarily 0? Give a proof (explanation) or counterexample. You may use a theorem proved in class.
   No, for example $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

   (b) (10 pts) Answer the same question, assuming now that A is symmetric.
   In this case, the answer is yes. We know by a theorem proved in class that A is diagonalizable: $A = SDS^{-1}$. Since D just has as its entries the eigenvalues of A, $D = 0$. Hence $A = 0$.

   However, if complex entries are allowed, the answer is yes, for example the matrix $\begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}$. For complex entries, for the theorem to be true, "symmetric" should be replaced by "hermitian."

5. (2 pts) Let P_2 be the vector space of polynomials of degree less than or equal to 2, and let $\mathcal{B} := (1, x, x^2)$ denote the standard ordered basis for P_2. Let $T : P_2 \to P_2$ denote the mapping sending f to $f' + f$.

   (a) (5 pts) Show that T is a linear transformatiion.
   This just says that if $f, g \in P_2$ and $a, b \in R$, then $(af + bg)' + (af + bg) = a(f' + f) + b(g' + g)$.

   (b) (5 pts) Find the matrix A for T with respect to the basis $\mathcal{B}$.
   The jth column of A is the column of coefficients of Tv_j. Thus

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (c) (5 pts) Find the eigenvectors and eigenvalues of A.
   Since A is upper triangular, its eigenvalues are the diagonal entries. Thus 1 is the only eigenvalue. Then

$$Eig_1(A) = NS \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = span \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

   (d) (5 pts) Is A diagonalizable? Explain.
   No, since there cannot be a basis of eigenvectors.