Do your calculations on the sheets and put a box around your answer where this makes sense.
Print your name and your TA’s name and section time here:

<table>
<thead>
<tr>
<th>Last Name</th>
<th>First</th>
<th>TA’s name</th>
<th>Section time</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Prob #</th>
<th>Max</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
Useful tips

FT pairs

<table>
<thead>
<tr>
<th>Signal $t \rightarrow x(t)$</th>
<th>$\omega \rightarrow X(\omega)$</th>
<th>$f \rightarrow X(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X(t)$</td>
<td>$2\pi x(-\omega)$</td>
<td>$x(-f)$</td>
</tr>
<tr>
<td>$x(t) \equiv 1$</td>
<td>$X(\omega) = 2\pi \delta(\omega)$</td>
<td>$X(f) = \delta(f)$</td>
</tr>
<tr>
<td>$x(t) = \delta(t)$</td>
<td>$X(\omega) \equiv 1$</td>
<td>$X(f) \equiv 1$</td>
</tr>
<tr>
<td>$x(t) = \text{sgn}(t)$</td>
<td>$X(\omega) = \frac{2 \delta}{j\omega}$</td>
<td>$X(f) = \frac{1}{j\pi f}$</td>
</tr>
<tr>
<td>$x(t) = u(t)$</td>
<td>$X(\omega) = \frac{1}{j\omega} + \pi \delta(\omega)$</td>
<td>$X(f) = \frac{1}{j2\pi f} + \frac{1}{2} \delta(f)$</td>
</tr>
<tr>
<td>$x(t) = \frac{1}{\pi t}$</td>
<td>$X(\omega) = -j \text{sgn}(\omega)$</td>
<td>$X(f) = -j \text{sgn}(f)$</td>
</tr>
<tr>
<td>$x(t) = \Pi(t) = 1$, $</td>
<td>t</td>
<td>\leq 1/2$, 0 else</td>
</tr>
<tr>
<td>$X_n \sum e^{jn\omega_0 t}$</td>
<td>$\sum X_n \delta(f - n f_0)$</td>
<td>$2\pi \sum \delta(\omega - n \omega_0)$</td>
</tr>
<tr>
<td>$\dot{x}(t)$</td>
<td>$-j \text{sgn}(f) X(f)$</td>
<td>$-j \text{sgn}(\omega) X(\omega)$</td>
</tr>
</tbody>
</table>

FT properties

| $x(at)$                  | $\frac{1}{|a|} X\left(\frac{\omega}{a}\right)$ | $\frac{1}{|a|} X\left(\frac{\omega}{a}\right)$ |
| $x \ast y$              | $X(f) Y(f)$                                   | $X(\omega) Y(\omega)$                           |
| $x(t) y(t)$             | $(X \ast Y)(f)$                               | $\frac{1}{2\pi} (X \ast Y)(\omega)$           |
| $X(t)$                  | $x(-f)$                                      | $2\pi x(-\omega)$                               |
| $e^{2\pi f_0 t} x(t)$   | $X(f - f_0)$                                  | $X(\omega - \omega_0)$                         |
| $\dot{x}(t)$            | $(j 2\pi f) X(f)$                             | $(j \omega) X(\omega)$                         |
| $\int_{-\infty}^{t} x(s) ds$ | $\frac{1}{j2\pi f} + \frac{1}{2} X(0) \delta(f)$ | $\frac{1}{j\omega} X(\omega) + \pi X(0) \delta(\omega)$ |

Parseval’s theorem.

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Trig identities

\[
\begin{align*}
\sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \\
\cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\
\sin x \sin y &= \frac{1}{2} [\cos(x - y) - \cos(x + y)] \\
\cos x \cos y &= \frac{1}{2} [\cos(x - y) + \cos(x + y)] \\
\sin x \cos y &= \frac{1}{2} [\sin(x - y) + \sin(x + y)]
\end{align*}
\]
1. **15 points** Find the FT of \( x \), and sketch the real and imaginary parts of \( X(\omega) \), where

\[
\forall t, \quad x(t) = \Pi(t) \ast \Pi(t) * \sum_{-\infty}^{\infty} \delta(t - 8n).
\]

Here \( \Pi(t) = 1 \) for \( |t| \leq 1/2 \) and 0, else. In your sketch carefully mark the relevant frequencies and magnitudes.

**Answer**

From Table above the FT of

\[
\Pi \leftrightarrow X_1: \omega \mapsto \frac{\sin(\omega/2)}{\omega/2}.
\]

From the Table above,

\[
t \mapsto \sum_{-\infty}^{\infty} \delta(t - 8n) \leftrightarrow X_2: \omega \mapsto \frac{2\pi}{8} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{8}).
\]

By the convolution property,

\[
x \leftrightarrow X(\omega) = [X_1(\omega)]^2 X_2(\omega)
\]

\[
= \left[ \frac{\sin(\omega/2)}{\omega/2} \right]^2 \frac{2\pi}{8} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{8})
\]

\[
= \frac{2\pi}{8} \sum_{k} \left[ \frac{\sin(2\pi k/16)}{2\pi k/16} \right]^2 \delta(\omega - \frac{2\pi k}{8})
\]

\( X \) is a real-valued function, its imaginary part is zero. A sketch of \( X \) is shown in figure 1.

---

Figure 1: Sketch of \( X \) in problem 1. Only the positive frequencies are shown, since \( X(\omega) = X(-\omega) \).
2. **15 points**

(a) Find and sketch the FT of

\[ x(t) = \left( \frac{\sin \pi t}{\pi t} \right)^2 e^{-j2\pi \times 10t}. \]

(b) Use Parseval’s theorem to find the energy in the signal \( x \).

**Answer**

From Table above,

\[ \Pi(t) \leftrightarrow \frac{\sin \omega / 2}{\omega / 2}. \]

From Table above,

\[ \frac{\sin \pi t}{\pi t} \leftrightarrow 2\pi \Pi(-\omega) = 2\pi \Pi(\omega) \]

From Table above, \([x(t)]^2 \leftrightarrow (1/2\pi)(X * X)(\omega)\), so

\[ \left( \frac{\sin \pi t}{\pi t} \right)^2 \leftrightarrow \frac{1}{2\pi} (2\pi)^2 (\Pi * \Pi)(\omega) = 2\pi (\Pi * \Pi)(\omega) \]

\( \Pi * \Pi \) has the triangle shaped graph shown in Figure 2.

(a) The FT of \( x \) is

\[ X(\omega) = 2\pi (\Pi * \Pi)(\omega - 2\pi \times 10) \]

and is sketched in Figure 2.

(b) By Parseval’s theorem the energy in \( x \) is

\[ \int |x(t)|^2 dt = \frac{1}{2\pi} \int |X(\omega)|^2 d\omega = \frac{2}{2\pi} \int_{0}^{2\pi} \left( \frac{\omega}{2\pi} \right)^2 d\omega = \frac{2}{3} \]
3. **10 points** The following statements are either TRUE or FALSE. If you believe a statement is true, outline a BRIEF PROOF. If you believe it is false, provide a BRIEF COUNTEREXAMPLE.

   (a) If \(x(t), t \in \text{Reals}\), is a real-valued signal, its Fourier transform \(X(f), f \in \text{Reals}\), is also real-valued.

   (b) If \(x(t), y(t), t \in \text{Reals}\), are real-valued signals and \((x * y)(t) = 0, \forall t \in \text{Reals}\), then either \(x\) or \(y\) is identically zero.

   (c) If \(x(t), t \in \text{Reals}\), is a real-valued, baseband signal with bandwidth \(W\) Hz, then the signal \(y(t) = x^4(t), t \in \text{Reals}\), has bandwidth at most \(4W\) Hz.

   (d) If \(x(t), t \in \text{Reals}\) is a real-valued, band-limited signal with bandwidth \(W\) Hz, then the signal \(y(t) = x(2t), t \in \text{Reals}\), has bandwidth \(W/2\) Hz.

   (e) If \(x, y\) are real-valued signals with bandwidth \(W_x, W_y\) Hz, respectively, then the signal \(x + y\) has bandwidth \(W_x + W_y\) Hz.

**Answer**

(a) **FALSE.** The function \(\forall t, x(t) = \text{sgn}(t)\) is real-valued, but its Fourier transform is \(\forall \omega, X(\omega) = \frac{2}{j \omega}\) which is not real-valued.

(b) **FALSE.** Take a band-limited signal \(x\) for example, \(x(t) = \sin(t)/t\) has bandwidth 1 radian/sec, so \(|X(\omega)| = 0, |\omega| > 1\). Now take \(y(t) = x(t) \cos(10t)\). Then \(Y(\omega) = 1/2[X(\omega - 10)+X(\omega+10)]\). It follows that \(X(\omega)Y(\omega) = 0\) for all \(\omega\). But then \((x * y)(t) = 0\) for all \(t\), even though neither \(x\) nor \(y\) is identically zero.

(c) **TRUE.** \(y \leftrightarrow Y = X * X * X * X\). If \(X(f) = 0\) for \(|f| > W\), then \(X * X * X * X(f) = 0\) for \(|f| > 4W\).

(d) **FALSE.** From the time scale property, \(Y(f) = 1/2 X(f/2)\). So the bandwidth of \(y\) is \(2W \neq W^2\) if \(W \neq 2\).

(e) **FALSE.** Take \(y = x\). Then \(x + y \leftrightarrow 2X\) has bandwidth \(W\).
4. **15 points** The following statements are either TRUE or FALSE. If you believe a statement is true, outline a BRIEF PROOF. If you believe it is false, provide a BRIEF COUNTEREXAMPLE.

(a) The system that takes as input a signal \( m \) and produces its Hilbert transform \( \hat{m} \) as output is an LTI system.

(b) The SSB-USB modulator which takes as input a signal \( m(t), t \in \text{Reals} \), and produces as output the modulated signal \( x(t), t \in \text{Reals} \), is a linear system.

(c) The narrow-band FM system which takes as input the continuous-time signal \( m \) and produces as output the modulated signal \( x \), is a linear system.

(d) The AM-DSB modulator is a time-invariant system.

(e) The signal \( \forall t, x(t) = \cos(2\pi f_c t + \cos(2\pi f_m t)) \) has infinite bandwidth.

(f) It is possible to recover the signals \( A \) and \( \theta \) from the narrowband signal \( \forall t, x(t) = A(t) \cos(2\pi f_c t + \theta(t)) \).

**Answer**

(a) **TRUE**, because \( \hat{m} = m * h \) where \( h \) is the impulse response of the Hilbert transform.

(b) **TRUE**, because

\[
\forall t, \quad x(t) = [m(t) + j\hat{m}(t)]e^{j\omega_c t},
\]

\[
= (m * (\delta + jh))(t)e^{j\omega_c t},
\]

(where \( h \) is as in part (a) and the operations above are linear.

(c) **FALSE**, because the signal \( x \) is

\[
\forall t, \quad x(t) = \cos 2\pi f_c t - m(t) \sin 2\pi f_c t,
\]

and this is not a linear relation: for example \( x \neq 0 \) even if \( m(t) \equiv 0 \).

(d) **FALSE**. The modulator is a linear, memoryless, time-varying system:

\[
x(t) = m(t) \cos 2\pi f_c t.
\]

(e) **TRUE**. The signal is periodic in \( t \) with period \( 1/f_m \) since \( x(t + 1/f_m) = \cos(2\pi f_m (t + 1/f_m)) = \cos(2\pi f_m t) \). It is also an even function of \( t \). Hence \( x \) has a Fourier series representation

\[
x(t) = \sum_{k=0}^{\infty} a_k \cos(k2\pi f_m t),
\]

which has infinite bandwidth.

(f) **TRUE**. Construct the signal \( z \) by

\[
\forall t, \quad z(t) = |x(t) + j\hat{x}(t)|e^{-j2\pi f_c t},
\]

where \( \hat{x} \) is the Hilbert transform of \( x \). Then \( z(t) = A(t)e^{j\theta(t)} \), so \( A(t) = |z(t)| \) and \( \theta(t) = \arg z(t) \).
5. **20 points** Figure 3 is a block diagram of vestigial sideband (VSB) modulation/demodulation. The

[Diagram of VSB modulation-demodulation scheme]

Figure 3: The VSB modulation-demodulation scheme of problem 5

baseband signal \( m \) has FT \( M \) as shown, with bandwidth \( B \) rad/sec. It modulates the carrier \( \cos(\omega_c t) \) \( (\omega_c >> B) \) to produce the signal \( u \), which is passed through the VSB filter, whose frequency response \( H(\omega) \) is shown. The result is the transmitted signal \( v \). The coherent receiver multiplies \( v \) by the carrier to produce \( w \), which is then passed through a low pass filter (LPF) to obtain the signal \( x \).

(a) Sketch the FT of \( u, v, \) and \( w \). Carefully mark relevant magnitudes and frequencies.

(b) Show that \( x = \frac{1}{4}m \) if the VSB filter satisfies

\[
H(\omega + \omega_c) + H(\omega - \omega_c) = 1, \text{ for } |\omega| \leq B.
\]

**Answer** We have

\[
U(\omega) = \frac{1}{2}[M(\omega - \omega_c) + M(\omega + \omega_c)], \quad V(\omega) = \frac{1}{2}H(\omega)[M(\omega - \omega_c) + M(\omega + \omega_c)]
\]

\[
W(\omega) = \frac{1}{2}[V(\omega - \omega_c) + V(\omega + \omega_c)]
\]

\[
= \frac{1}{4} \left[ H(\omega - \omega_c)\{M(\omega - 2\omega_c) + M(\omega)\} + H(\omega - \omega_c)\{M(\omega) + M(\omega + 2\omega_c)\} \right]
\]

\[
= \frac{1}{4}M(\omega)[H(\omega - \omega_c) + H(\omega + \omega_c)], \text{ for } |\omega| < B
\]

[Diagram of FTs for problem 5]
Figure 4 shows the FTs for (a). Part (b) follows from the last equation.
6. **10 points** Figure 5 is a block diagram of a digital communication system. The digital channel accepts at its input port any symbol from \{a, b, c, d, e\} and delivers it at its output port. The channel can accept one symbol every \(2 \mu\text{sec}\).

(a) What is the baud rate of the channel? What is its capacity in bits/sec?

(b) A binary source \(m\) produces data at \(1\)Mb/sec. (1 Mb is one million bits.) Is the rate of the source smaller than the capacity? If it is, construct a "coder" that maps the binary source \(m\) into a sequence of symbols \(x\), and a "decoder" that maps \(x\) into a binary sequence \(m'\) such that \(m' = m\).

**Answer**

(a) The baud rate is \(1/(2 \times 10^{-6}) = 500,000\) symbols/sec. The channel capacity is \(C = \log_2(5) \times 500,000\) bps.

(b) Yes. \(C > 10^6\), since \(\log_2(5) > \log_2(4) = 2\).

The coder should map pairs of bits into distinct symbols, the decoder should do the inverse. They are given by the following assignments:

Coder: \(00 \rightarrow a; 01 \rightarrow b; 10 \rightarrow c; 11 \rightarrow d\)

Decoder: \(a \rightarrow 00; b \rightarrow 01; c \rightarrow 10; d \rightarrow 11\)
7. **20 points** \( m \) is a complex-valued signal with bandwidth \( B_m \) rad/sec whose real and imaginary parts are \( m_1, m_2 \) respectively. Let \( M(\omega), M_1(\omega) \) and \( M_2(\omega) \) be the FT of \( m, m_1 \) and \( m_2 \), respectively.

(a) Find \( M_1 \) and \( M_2 \) in terms of \( M \). Show that the bandwidth of \( m_1, m_2 \) is at most \( B_m \).

(b) Design a modulation and demodulation scheme that can transmit \( m_1 \) and \( m_2 \) over a channel with bandwidth \( 2B_m \) centered at frequency \( \omega_c \) rad/sec.

(c) Give a brief mathematical argument to show that the transmitted signal is within the channel bandwidth, and that the receiver can recover both signals.

**Answer**

(a) \( M(\omega) = \int m(t)e^{-j\omega t}dt; M^*(\omega) = \int m^*(t)e^{j\omega t}dt; M^*(-\omega) = \int m^*(t)e^{-j\omega t}dt \). Since \( m + m^* = 2m_1 \) and \( m - m^* = 2jm_2 \),

\[
M(\omega) + M^*(-\omega) = \int [m(t) + m^*(t)]e^{-j\omega t}dt = 2M_1(\omega)
\]

\[
M(\omega) - M^*(-\omega) = \int [m(t) - m^*(t)]e^{-j\omega t}dt = 2jM_2(\omega)
\]

(b) The modulated signal is

\[
x(t) = m_1(t)\cos(\omega_c t) + m_2(t)\sin(\omega_c t).
\]

To recover the signals we use coherent demodulation. Multiply \( x \) by \( \cos(\omega_c t) \) and pass the product through a LPF with cutoff \( B_m \) to recover \( m_1 \); multiply \( x \) by \( \sin(\omega_c t) \) and pass the product through a LPF with cutoff \( B_m \) Hz to recover \( m_2 \).

(c) We have

\[
x(t) \leftrightarrow X(\omega) = M_1(\omega) + \frac{1}{2}\left[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)\right] + M_2(\omega)\frac{1}{2j}\left[\delta(\omega - \omega_c) - \delta(\omega + \omega_c)\right]
\]

\[
= \frac{1}{2}\left[M_1(\omega - \omega_c) + M_1(\omega + \omega_c)\right] + \frac{1}{2j}\left[M_2(\omega - \omega_c) - M_2(\omega + \omega_c)\right],
\]

which shows that \( |X(\omega)| = 0, ||\omega| - \omega_c| > B_m\).

To show that the demodulation scheme works:

\[
x(t)\cos(2\pi f_c t) = m_1(t)\cos^2(\omega_c t) + m_2(t)\sin(\omega_c t)\cos(\omega_c t)
\]

\[
= \frac{1}{2}[m_1(t) + m_1(t)\cos(2\omega_c t)] + \frac{1}{2}m_2(t)\sin(2\omega_c t)
\]

\[
\rightarrow \frac{1}{2}m_1(t) \text{ after passing through LPF},
\]

and similarly,

\[
x(t)\sin(\omega_c t) = m_1(t)\cos(\omega_c t)\sin(\omega_c t) + m_2(t)\sin^2(\omega_c t)
\]

\[
= \frac{1}{2}m_1(t)\sin(2\omega_c t) + \frac{1}{2}[m_2(t) - m_2(t)\cos(2\omega_c t)]
\]

\[
\rightarrow \frac{1}{2}m_2(t) \text{ after passing through LPF}.
\]