MATH 54: MIDTERM 1 SOLUTIONS

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3. True or False

Q3. *Select "true" (i.e., always true) or "false" (i.e., sometimes false) for each statement.*

Q3.1. *If the linear system $A\mathbf{x} = \mathbf{0}$ has at least one solution then $A\mathbf{x} = \mathbf{b}$ must have at least one solution.*

A3.1. False. The homogeneous system $A\mathbf{x} = \mathbf{0}$ always has at least the trivial solution ($\mathbf{x} = \mathbf{0}$) but of course we can find a matrix A and vector $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ is inconsistent. For example, set

$$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Q3.2. *If the linear system $A\mathbf{x} = \mathbf{0}$ has at most one solution then $A\mathbf{x} = \mathbf{b}$ has at most one solution.*

A3.2. True. Suppose $\mathbf{v}, \mathbf{w}$ are two distinct solutions to $A\mathbf{x} = \mathbf{b}$. Then $A(\mathbf{v} - \mathbf{w}) = A\mathbf{v} - A\mathbf{w} = \mathbf{b} - \mathbf{b} = \mathbf{0}$ and so $\mathbf{v} - \mathbf{w} \neq \mathbf{0}$ is a nontrivial solution to $A\mathbf{x} = \mathbf{0}$. Thus, along with the trivial solution, $A\mathbf{x} = \mathbf{0}$ has two distinct solutions.

Q3.3. *If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$ are vectors such that $\{\mathbf{x}, \mathbf{y}\}$ are linearly independent and $\{\mathbf{y}, \mathbf{z}\}$ are linearly independent then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ must be linearly independent.*

A3.3. False. Consider the vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Any set containing two of these vectors is linearly independent (because no two are parallel), but the three vectors are linearly dependent because $\mathbf{x} + \mathbf{y} = \mathbf{z}$.

Q3.4. *If $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$ are linearly independent vectors and $\mathbf{v}_3 \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must be linearly independent.*

A3.4. True. Suppose $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly independent and $\mathbf{v}_3 \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, but for contradiction that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly dependent. Then there are a_1, a_2, a_3 not all zero so that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$. We must have $a_3 = 0$, since otherwise $\mathbf{v}_3 = -(a_1/a_3)\mathbf{v}_1 - (a_2/a_3)\mathbf{v}_2 \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. But now $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = \mathbf{0}$, a nontrivial linear dependence among $\{\mathbf{v}_1, \mathbf{v}_2\}$. This contradicts the assumption that $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly independent. We conclude that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent.

An alternate argument is simply to apply Theorem 7 in the book: if the vectors were linearly dependent then either $v_2 \in span\{v_1\}$ or $v_3 \in span\{v_1, v_2\}$, both of which are ruled out by the hypothesis.

Q3.5. *If R is the reduced row echelon form of an $m \times n$ matrix A and $A\mathbf{x} = \mathbf{b}$ is consistent for some vector $\mathbf{b} \in \mathbb{R}^m$, then $R\mathbf{x} = \mathbf{b}$ must be consistent.*

A3.5. False. When we row-reduce the augmented matrix $[A : \mathbf{b}]$ to determine whether the system $A\mathbf{x} = \mathbf{b}$ is consistent, we end up with the echelon form $[R : \mathbf{c}]$, where in general $\mathbf{c} \neq \mathbf{b}$ (the row operations affect the augmented column). Here's an example that shows the proposition failing:

$$[A : \mathbf{b}] = \left[\begin{array}{c:c} 0 & 0 \\ 1 & 1 \end{array}\right], \qquad \text{but} \qquad [R : \mathbf{b}] = \left[\begin{array}{c:c} 1 & 0 \\ 0 & 1 \end{array}\right].$$

$[A : \mathbf{b}]$ represents a consistent system (with unique solution $\mathbf{x} = \begin{bmatrix} 1 \end{bmatrix}$), but $[R : \mathbf{b}]$ represents an inconsistent system (it's the same system from the counterexample to **3.1**).

Q3.6. *If A, B are $n \times n$ matrices then* $\det(A+B) = \det(A) + \det(B)$.

A3.6. False. Let $A = B = I$, the $n \times n$ identity matrix, for $n \geqslant 2$. Then $\det(A) + \det(B) = 1 + 1 \neq 2^n = \det(A+B)$.

Q3.7. *If A is a square matrix such that A^2 is invertible, then A must be invertible.*

A3.7. True. Let $B = (A^2)^{-1}$. I claim that A is invertible wtih inverse AB. We have

$$A(AB) = (A^2)B = (A^2)(A^2)^{-1} = I.$$

This shows that A is onto (it has a right inverse). For square matrices, this is equivalent to having a left inverse, so A is invertible.

An alternate argument is that $0 \neq \det(A^2) = \det(A)^2$ so $\det(A) \neq 0$ and A must be invertible by the invertible matrix theorem.

Q3.8. *If $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $\mathbb{R}^3$ and A, B are 4×3 matrices satisfying $A\mathbf{b}_i = B\mathbf{b}_i$ for $i = 1, 2, 3$, then $A = B$.*

A3.8. True. Since b_1, b_2, b_3 is a basis of $\mathbb{R}^3$, we can express the standard basis vector e_1 as a linear combination $e_1 = c_1b_1 + c_2b_2 + c_3b_3$. Thus,

$$Ae_1 = A(c_1b_1 + c_2b_2 + c_3b_3) = c_1Ab_1 + \ldots c_3Ab_3 = c_1Bb_1 + \ldots + c_3Bb_3 = B(c_1b_1 + \ldots + c_3b_3) = Be_1.$$

Repeating the argument with e_2, e_3 shows that $A = B$.

Q3.9. *If H is a subspace of $\mathbb{R}^5$, $\mathbf{v}_1, \ldots, \mathbf{v}_4 \in H$, and $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ are linearly independent, then* $\dim(H) \geqslant 4$.

A3.9. True. Suppose $\dim(H) = k$ and $b_1, \ldots, b_k$ is a basis of H. Let B be the $5 \times k$ matrix with $b_1, \ldots, b_k$ as its columns. By the spanning property, we can write $v_i = Bx_i$ for some vectors $x_i \in \mathbb{R}^k$. If $k < 4$ then by the too many vectors theorem, there must be a nontrivial linear dependence $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0$. Multiplying by B, we find that $c_1v_1 + \ldots + c_4v_4 = 0$, contradicting our assumption that $v_1, \ldots, v_4$ are linearly independent. Thus $k \geqslant 4$ and $\dim(H) \geqslant 4$.

Q3.10. *If H is a subspace of $\mathbb{R}^5$ and* $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_4\} = H$, *then* $\dim(H) \geqslant 4$.

A3.10. False. In this case, taking $\mathbf{v}_1 = \cdots = \mathbf{v}_4 = \mathbf{0}$, we have $H = \{\mathbf{0}\}$ which does not have dimension $\geqslant 4$. *On the other hand, it is true that if a subspace is spanned by k vectors, then its dimension is $\leqslant k$.*

4. Linear Systems

Q4. *Give an example of each of the following, explaining why it has the required property, or explain why no example exists.*

Q4.1. *Two vectors $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^3$ and a 3×3 matrix A such that the linear system $A\mathbf{x} = \mathbf{b}_1$ has exactly one solution and the linear system $A\mathbf{x} = \mathbf{b}_2$ is inconsistent.*

A4.1. This is impossible. If there is a vector $\mathbf{b}_1$ for which $A\mathbf{x} = \mathbf{b}_1$ has exactly one solution, then $A\mathbf{x} = \mathbf{0}$ has exactly one solution, in which case A represents a one-to-one transformation $\mathbb{R}^3 \to \mathbb{R}^3$. Since A is square, this means A is also onto, and therefore that there can be no $\mathbf{b}_2 \in \mathbb{R}^3$ for which $A\mathbf{x} = \mathbf{b}_2$ is inconsistent.

Q4.2. *Two nonzero 2×2 matrices A and B such that*

$$(A+B)^2 = A^2 + B^2.$$

A4.2. Consider the following two matrices.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We can check that $A^2 = A$, $B^2 = B$, and $AB = BA = 0$ (the zero matrix). So $(A+B)^2 = A^2 + AB + BA + B^2 = A^2 + B^2$, as desired.

Q4.3. *An onto linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \textit{and} \qquad T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

A4.3. This is impossible. By adding the two given equations and applying linearity, one has

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 0$$

Thus, $T(x) = 0$ does not have a unique solution so T is not one to one. But since $m = n$ this means that T cannot be onto.

Q4.4. *A 2×4 matrix A such that* $\mathrm{Null}(A)$ *has dimension equal to* 3*.*

A4.4. This is absolutely possible. One example is

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The columnspace of this matrix is $\mathrm{span}\{(1,0)\}$, so $\dim \mathrm{Col}(A) = 1$ and therefore $\dim \mathrm{Null}(A) = 4 - 1 = 3$.

5. Outside Span

Q5. *Consider the following vectors in $\mathbb{R}^3$*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Find the first vector in this list which is not in the span of the other vectors. Explain your reasoning.

A5. The statement $\mathbf{v}_i \notin \mathrm{span}\{\mathbf{v}_j \mid j \neq i\}$ is equivalent to saying that $a_i = 0$ in *every* linear dependence

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 = \mathbf{0}$$

(otherwise, we could perform the same manipulation we discussed in **3.4**). We are looking for the smallest i such that every such dependence atisfies $a_i = 0$.

The set of all linear dependencies between the given vectors is given by the null space of the matrix with these vectors as its columns. Let's row-reduce the matrix whose columns are $\{\mathbf{v}_i\}$.

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 0 & 3 & -1 \\ 3 & 2 & 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -2 & 5 & -5 \\ 0 & -1 & 5 & -5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Notice that the matrix above has one free variable. That means its nullspace equals $\mathrm{span}\{(1,0,-1,-1)\}$, and all linear dependencies are multiples of:

$$\mathbf{v}_1 + 0\,\mathbf{v}_2 - 1\,\mathbf{v}_3 - 1\,\mathbf{v}_4 = \mathbf{0},$$

We conclude that $\mathbf{v}_2$ is the first vector above not in the span of the others.

6. Inverse

Q6. *Consider the matrix*

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}.$$

Q6a. *Is A invertible? If so, compute its inverse. If not, explain why.*

A6a. We can compute the determinant by cofactor expanding:

$$\det(A) = \det \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} = 0 \det \begin{bmatrix} 0 & 3 \\ -3 & 8 \end{bmatrix} - 1 \det \begin{bmatrix} 1 & 3 \\ 4 & 8 \end{bmatrix} + 2 \det \begin{bmatrix} 1 & 0 \\ 4 & -3 \end{bmatrix} = -(-4) + 2(-3) = -2.$$

Thus A is invertible, since its determinant is nonzero. We can compute the inverse by row reduction.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right].$$

We conclude that

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

Q6b. *Find a solution $\mathbf{x} \in \mathbb{R}^3$ to the linear system*

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

A6b. The quickest way to produce this solution is simply to multiply $(1, 0, 1)$ by A^{-1}, which we computed above. The solution is

$$\mathbf{x} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ 2 \end{bmatrix}$$

Q6c. *Is the solution you found above unique? Explain why or why not.*

A6c. Yes – since A is invertible, for any $\mathbf{b} \in \mathbb{R}^3$, the system $A\mathbf{x} = \mathbf{b}$ is consistent and has the unique solution $\mathbf{b} = A^{-1}\mathbf{x}$.

7. Determinant

Q7. *Find the determinant of the matrix*

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 & 5 \\ 1 & 0 & 3 & 4 & 5 \\ -1 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

A7. Notice that by replacing R_3 by $R_2 + R_3$, and then swapping R_1 and R_2, we obtain the matrix

$$B = \begin{bmatrix} 1 & 0 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 6 & 8 & 10 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

Since we obtained B from A by performing a row addition and a single rowswap, $\det(B) = -\det(A)$. And since B is upper-triangular, its determinant is the product of the diagonal entries: $\det(B) = 1\cdot2\cdot6\cdot4\cdot4 = 192$. Thus $\det(A) = -192$.

8. Both Subspaces

Q8. *Consider the matrices*

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ -1 & 2 & -1 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -1 & -3 & 1 \end{bmatrix}.$$

Q8a. *Find a nonzero vector* $\mathbf{v} \in \mathbb{R}^3$ *which is an element of both the subspaces* $\mathrm{Null}(A) \subset \mathbb{R}^3$ *and* $\mathrm{Col}(B) \subset \mathbb{R}^3$. *Explain your reasoning.*

A8a. We first find a basis for the nullspace of A. Row-reducing:

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ -1 & 2 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathrm{Null}(A) = \mathrm{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We'll call the two vectors above $\mathbf{v}_1, \mathbf{v}_2$, and we'll call the columns of B $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. We now seek a vector that is both a linear combination of the $\{\mathbf{v}_i\}$ and of the $\{\mathbf{w}_i\}$. One way to do this is to find a linear dependence $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{w}_1 + a_4\mathbf{w}_2 + a_5\mathbf{w}_3$. Then the two vectors $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = -(a_3\mathbf{w}_1 + a_4\mathbf{w}_2 + a_5\mathbf{w}_3)$ are the same; one in the nullspace of A, and the other in the columnspace of B. Provided a_1, a_2 are not both zero, that vector is guaranteed nonzero since the $\{\mathbf{v}_i\}$ are linearly independent. So we want to solve $C\mathbf{x} = \mathbf{0}$, where

$$C = \begin{bmatrix} 2 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & -1 & -3 & 1 \end{bmatrix}.$$

Row-reducing, we have

$$\begin{bmatrix} 2 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & -1 & -3 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & -1 & -3 & 1 \\ 0 & -1 & 1 & 3 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & -1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This linear system has three free variables (w_1, w_2, w_3), so actually *any* nonzero vector in $\mathrm{Col}(B)$ is a valid solution to the problem! For example, the vectors $(2,1,0), (1,1,1), (-1,-2,-3)$ and any nonzero linear combinations of them are valid answers. In fact, in this problem, we have $Nul(A) = \mathrm{Col}(B)$.

Q8b. *Are the columns of the product AB linearly independent? Explain why or why not based on your answer to **8a**, without doing any matrix multiplication.*

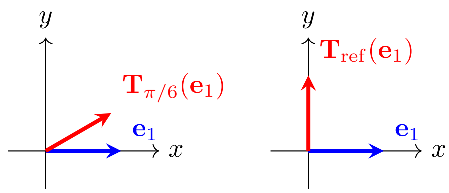
A8b. The columns of AB are not linearly independent. Since we found a nonzero vector $(1, 1, 1) \in \text{Null}(A) \cap \text{Col}(B)$, there is some vector $\mathbf{x}$ so that $B\mathbf{x} = (1, 1, 1)$ and $A(B\mathbf{x}) = \mathbf{0}$ (in fact, $\mathbf{x} = (0, 0, 1)$; this means that the third column of AB is zero).

9. Rotation and Reflection

Q9. *Let $T_{\pi/6} : \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear transformation which rotates a vector in $\mathbb{R}^2$ counterclockwise by $\pi/6$ radians. Let $T_{\text{ref}} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which reflects a vector $\mathbf{x} = (x_1, x_2)$ across the line $x_1 = x_2$.*

Q9a. *Sketch a cartoon illustrating what these linear transformations do to the vector $\mathbf{e}_1 = (1, 0)$.*

A9a. Below are cartoons of these transformations acting on $\mathbf{e}_1$.



Q9b. *Find the standard matrices for $T_{\pi/6}$ and T_{ref}.*

A9b. We find the standard matrix for $T_{\pi/6}$ using trigonometry. Rotating the point $(1, 0)$ counterclockwise by $\pi/6$ yields $(\cos(\pi/6), \sin(\pi/6)) = (\sqrt{3}/2, 1/2)$. Rotating $(0, 1)$ counterclockwise by $\pi/6$ yields $(-\sin(\pi/6), \cos(\pi/6)) = (-1/2, \sqrt{3}/2)$. So the standard matrix A for $T_{\pi/6}$ is

$$A = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}.$$

To find the standard matrix for T_{ref}, note that reflecting through the line $x_1 = x_2$ has the effect of swapping the coordinates of a point. That is, $T_{\text{ref}}(\mathbf{e}_1) = \mathbf{e}_2$ and $T_{\text{ref}}(\mathbf{e}_2) = \mathbf{e}_1$. Therefore the standard matrix B of T_{ref} is

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Q9c. *Find a nonzero vector $v \in \mathbb{R}^2$ such that $(T_{\pi/6} \circ T_{\text{ref}})(\mathbf{v}) = (T_{\text{ref}} \circ T_{\pi/6})(\mathbf{v})$ or explain why no such vector exists.*

A9c. We compute

$$\begin{aligned} AB - BA &= \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \\ &= \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} - \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

This matrix in invertible. Thus there is no nonzero vector $\mathbf{v}$ with $(AB - BA)\mathbf{v} = 0$, equivalently $AB\mathbf{v} = BA\mathbf{v}$, or equivalently, with $(T_{\pi/6} \circ T_{\text{ref}})(\mathbf{v}) = (T_{\text{ref}} \circ T_{\pi/6})(\mathbf{v})$ since matrix multiplication corresponds to composition of linear transformations..