Physics 89 Midterm Exam
Thursday, 10/15/2020, 12:40-2:10pm (PT)

Name or SID: ____________

Instructions

1. Please submit the exam using **Gradescope**.
2. Please solve all 3 problems below, and write your final answers in the space provided.
3. You can use the additional space at the end of each problem to show your work.
4. You can use the sheets at the end of the exam if you need extra space, or provide your own pages.
5. You are required to work on the exam on your own. In particular, collaboration or consultation with others is not permitted.
6. The maximal score is 102.

The Lecture zoom link will be open during the exam, and you can ask questions via the **chat**, or by email.

Good luck!

UC Berkeley's Honor Code

"As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others."

- I alone am taking this exam.
- I will not receive assistance from anyone while taking the exam nor will I provide assistance to anyone while the exam is still in progress.
- Other than with the instructor and GSI, I will not have any verbal, written, or electronic communication with anyone else while I am taking the exam or while others are taking the exam.

Problem 1 [34pts] – short answers

(a) Define the two functions $f(x)$ and $g(x)$ by

$$f(x) = \frac{1}{x^2+6x+5}, \qquad g(x) = \frac{1}{x^2+4x+5}.$$

The two Taylor series

$$\begin{aligned} f(x) &= \frac{1}{x^2+6x+5} = \tfrac{1}{5} - \tfrac{6}{25}x + \tfrac{31}{125}x^3 + \cdots \\ g(x) &= \frac{1}{x^2+4x+5} = \tfrac{1}{5} - \tfrac{4}{25}x + \tfrac{11}{125}x^3 + \cdots \end{aligned}$$

turn out to have different segments of convergence. The series for $g(x)$ converges for $|x| < \sqrt{5}$ and doesn't converge for $|x| > \sqrt{5}$, while the series for $f(x)$ converges for $|x| < 1$ and doesn't converge for $|x| > 1$.

Can you explain this fact using complex numbers?

[It is not important for this problem what happens at $x = \pm 1$ for $f(x)$ and $x = \pm\sqrt{5}$ for $g(x)$.]

(b) Recall our matrix notation for a linear system of equations in 3 variables X, Y, Z:

$$\left.\begin{aligned} aX + bY + cZ &= p \\ dX + eY + fZ &= q \\ gX + hY + lZ &= s \end{aligned}\right\} \Longrightarrow \qquad \mathbf{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & l \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a & b & c & p \\ d & e & f & q \\ g & h & l & s \end{pmatrix}.$$

Suppose after some **row-operations** we bring $\mathbf{A}$ to the form

$$\xrightarrow{\text{Row operations}} \begin{pmatrix} 1 & 0 & 4 & 5 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & T \end{pmatrix}.$$

For which value(s) of T, if any, will there be a unique solution for X, Y, Z?

For which value(s) of T, if any, will there be no solutions?

For which value(s) of T, if any, will there be more than one solution?

Problem 2 [34pts]

For each of the two differential equations,

$$y'(t) - y(t) = \cos(2t), \qquad y'(t) - y(t) = \sin(2t),$$

find a solution $y(t)$ using complex number methods.

Note: $y(t)$ is an unknown function of t that you need to find, and $y'(t)$ is its (unknown) derivative.

Problem 3 [34pts]

Given the matrix below

$$\mathbf{M} = \begin{pmatrix} 1 & -2 \\ -4 & 3 \end{pmatrix},$$

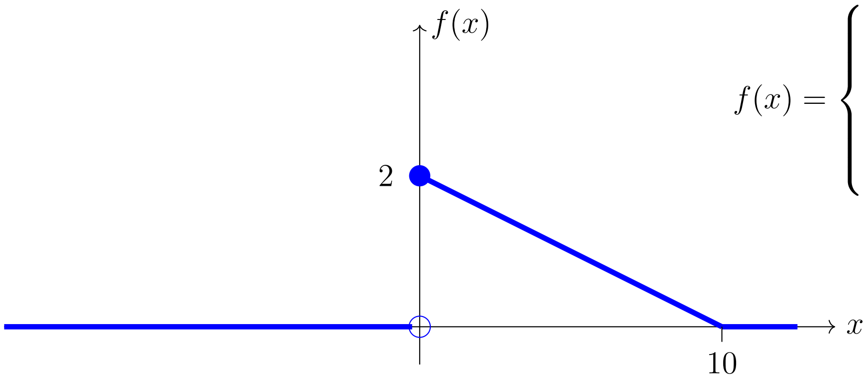
solve the following problems:

(a) Find all the eigenvalues of $\mathbf{M}$.

(b) For each eigenvalue from part (a), find a corresponding eigenvector.

(c) Find a matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{MC} = \mathbf{CD}$.

(d) Apply the function whose graph is depicted below to the matrix $\mathbf{M}$.
In other words, what is $f(\mathbf{M})$?



$$f(x) = \begin{cases} 0 & \text{for } x < 0; \\ 2 - \frac{1}{5}x & \text{for } 0 \le x \le 10; \\ 0 & \text{for } x > 10; \end{cases}$$

Solution to Problem 1

(a) We consider $f(z)$ and $g(z)$ as functions of a complex variable, and we look for the poles. We solve

$$f(z) = \infty \Longrightarrow z^2 + 6z + 5 = 0 \Longrightarrow z = -1, -5$$

So, the poles of $f(z)$ are at -1 and -5, and the one closest to the origin is -1. It's distance from the origin is 0, so the radius of convergence is 1.

Next, we solve

$$g(z) = \infty \Longrightarrow z^2 + 4z + 5 = 0 \Longrightarrow z = \frac{-4 \pm \sqrt{4^2 - 20}}{2} = -2 \pm i$$

So the poles are at $-2 \pm i$. Their distance from the origin is

$$|-2 \pm i| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$$

So the radius of convergence is $\sqrt{5}$.

(b) There are $n = 3$ unknowns. If $T = 0$ then the ranks of $\mathbf{A}$ and $\mathbf{M}$ are both $2 < n$ so there is an infinite number of solutions. If $T \neq 0$ the rank of $\mathbf{A}$ is 3 while the rank of $\mathbf{M}$ is 2, so there is no solution. There are no values of T for which there is a unique solution.

Problem 2 [34pts]

For each of the two differential equations,

$$y'(t) - y(t) = \cos(2t), \qquad y'(t) - y(t) = \sin(2t),$$

find a solution $y(t)$ using complex number methods.
Note: $y(t)$ is an unknown function of t that you need to find, and $y'(t)$ is its (unknown) derivative.

We combine the two equations and look for a complex solution $z(t)$ to

$$z'(t) - z(t) = \cos 2t + i \sin 2t = e^{2it}$$

We look for a solution of the form $z(t) = z_0 e^{2it}$. Then

$$z'(t) - z(t) = 2iz(t) - z(t) = (2i-1)z(t)$$

So,

$$z(t) = \frac{e^{2it}}{2i-1} = \frac{\cos 2t + i \sin 2t}{2i-1} = \frac{(\cos 2t + i \sin 2t)(-2i-1)}{(2i-1)(-2i-1)} = \frac{(2\sin 2t - \cos 2t) + i(-\sin 2t - 2\cos 2t)}{5}$$

So

$$y'(t) - y(t) = \cos(2t) \qquad \text{has solution } y(t) = \frac{2}{5}\sin 2t - \frac{1}{5}\cos 2t$$

and

$$y'(t) - y(t) = \sin(2t) \qquad \text{has solution } y(t) = -\frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t$$

Problem 3 [34pts]

(a)

$$= \det(\mathbf{M} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -2 \\ -4 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = -5 - 4\lambda + \lambda^2 \Longrightarrow \lambda = -1, 5/$$

(b) Set

$$\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$

For $\lambda = -1$ we get

$$0 = (\mathbf{M} - \lambda\mathbf{I})\mathbf{r} = \begin{pmatrix} 2 & -2 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Longrightarrow 0 = 2x - 2y = -4x + 4y$$

So we can take

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{as eigenvector for } \lambda = -1.$$

For $\lambda = 5$ we get

$$0 = (\mathbf{M} - \lambda\mathbf{I})\mathbf{r} = \begin{pmatrix} -4 & -2 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Longrightarrow 0 = -4x - 2y = -4x - 2y$$

So we can take

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad \text{as eigenvector for } \lambda = 5.$$

(c) Collecting the eigenvectors and eigenvalues we get

$$\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

Alternatively, the columns of $\mathbf{C}$ could be in the opposite order, so long as the diagonal elements of $\mathbf{D}$ are swapped accordingly.

(d)

$$f(\mathbf{M}) = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} f(-1) & 0 \\ 0 & f(5) \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}^{-1}$$

We note that $\det \mathbf{C} = 3$ and we calculate (using cofactors)

$$\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

Also,

$$f(-1) = 0, \qquad f(5) = 1$$

so

$$f(\mathbf{M}) = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1/3 & 1/3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$$