Check the units of your results.

Closed book, closed notes.

No calculators.

Leave pack, books, and electronic devices (e.g. cell phones) in isle.

Take off caps or hats.

Copy your answers into marked boxes on exam sheets.

Simplify numerical and algebraic results as much as possible.

Be kind to the graders and write legibly. No credit for illegible results.

No credit for multiple differing answers to the same question.

The UC rules on dishonesty apply.
1. [25 points] A capacitor $C_1$ is used to power a model airplane, represented in the circuit diagram below by resistor $R_1$. Initially the capacitor is charged to voltage $V_1$. Calculate the fraction $r$ of the initial energy remaining on the capacitor when the current $i_1$ has decreased to half its initial value.

\[ r = \frac{1}{4} \]

The energy stored in a capacitor is given by

\[ E_{\text{stored}} = \frac{1}{2} C V^2 \]

\[ E_0 = \frac{1}{2} C_1 V_0^2 = \frac{1}{2} C_1 V_1^2 \]

\[ E_f = \frac{1}{2} C_1 V_f^2 = \frac{1}{2} C_1 \left( \frac{i_1 R_1}{2} \right)^2 = \frac{1}{2} C_1 \left( \frac{i_1 R_1}{2} \right)^2 = \frac{1}{2} C_1 \left( \frac{V_f}{2} \right)^2 \]

\[ r = \frac{E_f}{E_0} = \frac{\frac{1}{2} C_1 \left( \frac{V_f}{2} \right)^2}{\frac{1}{2} C_1 V_1^2} = \frac{1}{4} \]
2. [25 points] In the circuit below $C_1$ represents a touch sensor. The comparator controls the position of the switch as follows: Whenever $V_o$ reaches the value $V_{ref}$, the switch is set to position $a$. After $V_o$ drops to zero, the switch is set to position $b$. Touch is detected by measuring the frequency $f_0$ at which the switch position changes. Derive an analytical expression for $f_0$. Assume that the operational amplifier is ideal. Suggestion: sketch $v_o(t)$ and mark the knowns and unknowns in the graph.

\[
f_0 = \frac{I_1 \cdot I_2}{C \cdot V_{ref} \cdot (I_1 + I_2)}
\]

The op-amp is in negative feedback. $v_p = v_n = 0$ $V_p = V_n = 0$ V

Let's define the current and voltage through $C_1$ as $\frac{e(t)}{C} + V_c(t)$

\[
I = \frac{C \cdot dV}{dt} \Rightarrow V_c(t) = \frac{1}{C} \int_0^t e(x) \, dx + V_c(0)
\]

**Charge Cycle (switch is set to b)**

\[
V_{ref} = \frac{1}{C} \int_0^{T_a} e(x) \, dx + V_c(0)
\]

$T_a = \frac{C \cdot V_{ref}}{I_2}$

**Discharge Cycle (switch is set to a)**

\[
0 = \frac{1}{C} \int_0^{T_b} e(x) \, dx + V_c(t_n)
\]

$T_b = \frac{C \cdot V_{ref}}{I_1}$

\[
T = T_a + T_b \Rightarrow f = \frac{1}{T_a + T_b}
\]

\[
f = \frac{1}{T_a} + \frac{C \cdot V_{ref}}{I_2} = \frac{I_1 \cdot I_2}{C \cdot V_{ref} \cdot (I_1 + I_2)}
\]

Note: $C_1$ charges through $I_1$ and $I_2$. $C_1$ discharges through $I_2$. 

3. [25 points] The circuit below shows a simple model of a wired Ethernet connection. $V_1$ is the signal source and $R_1$ represents its impedance. The cable is modeled by capacitor $C_2$ and $R_2$ represents the receiver's input impedance. The capacitor $C_1$ has been added to enable higher frequency transmission at low error rates.

Derive an expression for $C_1$ as a function of $R_1$, $R_2$, and $C_2$ for which $\frac{V_2(\omega)}{V_1(\omega)}$ has a constant value that is independent of frequency.

Note: This technique is used in many transceivers including 100 Mbit Ethernet.

![Circuit Diagram]

\[ C_1 = \frac{C_2 R_2}{R_1} \]

**Short Solution**

The impedance at low frequencies should equal the impedance at high frequencies. Since the caps are open at low freqs + dies shorts at high freqs

\[ Z(\omega) = Z(\text{hi}) \]

\[ \frac{1}{R_1 + R_2} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} \]

\[ \frac{R_2}{R_1 + R_2} = \frac{C_1}{C_1 + C_2} \]

\[ R_2 (C_1 + C_2) = C_1 (R_1 + R_2) \]

\[ C_1 = \frac{C_2 R_2}{R_1} \]
"Long" Solution

Let's find the transfer function \( \frac{v_2(s)}{v_1(s)} \)

\[
\frac{v_2(s)}{v_1(s)} = \frac{Z_2(s)}{Z_1(s)} = \frac{Z_2(s)}{R_2 + \frac{1}{sC_2}} = \frac{Z_2(s)}{R_2 + \frac{1}{sC_2}} \frac{R_1(s)}{R_1 + sR_1C_1}
\]

\[
= \frac{R_2}{R_1 + R_2} \frac{(1 + sR_1C_1)}{1 + sR_2C_2}
\]

\[
= \frac{R_2}{R_1 + R_2} \left( 1 + \frac{sR_1C_1}{1 + sR_2C_2} \right)
\]

In order for the transfer to be independent of frequency, the coefficients of \( s \) in \( C_1 \) and \( C_2 \) should be equal.

\[
R_1C_1 = \frac{R_1R_2}{R_1 + R_2} \quad (C_1 + C_2)
\]

\[
R_1C_1 (R_1 + R_2) = R_1R_2 (C_1 + C_2)
\]

\[
C_1 \frac{R_1}{R_1 + R_2} \geq C_2 \frac{R_2}{R_1 + R_2}
\]

\[
C_1 = \frac{C_2R_2}{R_1}
\]
4. [25 points] An audio system suffers from high frequency interference. Design a circuit consisting of a resistor $R$ and capacitor $C$ that passes low frequency signals ($\left| \frac{V_o(j\omega)}{V_i(j\omega)} \right|_{\omega=0} = 1$) and attenuates high frequency signals ($\left| \frac{V_o(j\omega)}{V_i(j\omega)} \right|_{\omega>0} < 1$).

(a) Draw a diagram of a circuit with these characteristics, consisting of $R$ and $C$. Clearly mark the input and output voltages $V_i$ and $V_o$ with plus and minus signs.

(b) Derive a symbolic expression for the value of $C$ as a function of $R$ that results in 26 dB attenuation at a given frequency $f_0$.

Hint: draw the Bode plot (piece-wise linear approximation of magnitude response) of the circuit. Mark what is known and unknown in the graph.

$$C = \frac{10}{\pi f_0 R} \quad \text{(Assume } f_0 \text{ is in Hz)} \quad \Rightarrow \quad C = \frac{20}{f_0 R} \quad \text{(Assume } f_0 \text{ is in rad/s)}$$

The transfer function of a first-order system is given by $H(j\omega) = \frac{1}{1+j\omega RC} = \frac{1}{1+j\omega wp}$. If $\omega < \omega_p$ (large), the imaginary term in the denominator dominates, and we have $H(j\omega) \approx \frac{1}{j\omega wp}$.

The magnitude in decibels is given by $20 \log_{10} |H(j\omega)|$.
Prop 4 cont.
The gain difference at a \( w_0 \) (frequency) 10 times \( wp \) is:

\[
\text{Gain}_{dB} (10\ wp) - \text{Gain}_{dB} (wp) = -20 \log_{10} (10\ wp) + 20 \log_{10} (wp) \\
= -20 \log_{10} (10) - 20 \log_{10} (wp) + 20 \log_{10} (wp) \\
= -20 \log_{10} (10) = -20\ dB \text{ (at 10 times wp)}
\]

The gain difference at a \( w_0 \) 2 times \( wp \) is:

\[
\text{Gain}_{dB} (2\ wp) - \text{Gain}_{dB} (wp) = -20 \log_{10} (2\ wp) + 20 \log_{10} (wp) \\
= -20 \log_{10} (2) - 20 \log_{10} (wp) + 20 \log_{10} (wp) \\
= -20 \log_{10} (2) = -6\ dB \text{ (at 2 times wp)}
\]

Now for our problem, we have 26 dB attenuation:

\[
26\ dB = 20\ dB + 6\ dB \uparrow \uparrow \text{10x } 2x \\
= \text{Gain}_{dB} (10\ wp) + \text{Gain}_{dB} (2\ wp) \\
= \text{Gain}_{dB} (20\ wp)
\]

\( w_0 = 20\ wp = 20 \cdot \frac{1}{RC} \)

\[C = \frac{20}{R \cdot w_0} \]

Since \( w_0 = 2\pi f_0 \)

\[C = \frac{20}{2\pi f_0 R} = \frac{10}{\pi f_0 R} \]