Math 1B — UCB, Fall 2019 — M. Christ

Midterm Exam 1 Solutions

There were two versions of the exam. These are solutions to version A.

(1a) Evaluate $\int \sec^7(x)\tan^3(x)\,dx$

Solution. Substitute $u = \sec(x)$ and $du = \tan(x)\sec(x)$.

$$\begin{aligned}
&= \int \sec^7(x)(\sec^2(x)-1)\tan(x)\,dx \\
&= \int \sec^6(x)(\sec^2(x)-1)\ \sec(x)\tan(x)\,dx \\
&= \int u^6(u^2-1)\,du \\
&= \int (u^8-u^6)\,du \\
&= \tfrac{1}{9}u^9 - \tfrac{1}{7}u^7 + C \\
&= \tfrac{1}{9}\sec^9(x) - \tfrac{1}{7}\sec^7(x) + C
\end{aligned}$$

□

(1b) $\int x^{1/2}\ln(x)\,dx$

Solution. IBP:

$$\begin{aligned}
&= \tfrac{2}{3}x^{3/2}\ln(x) - \tfrac{2}{3}\int x^{3/2}\,x^{-1}\,dx \\
&= \tfrac{2}{3}x^{3/2}\ln(x) - \tfrac{2}{3}\int x^{1/2}\,dx \\
&= \tfrac{2}{3}x^{3/2}\ln(x) - \tfrac{4}{9}x^{3/2} + C.
\end{aligned}$$

□

Alternative solution. Substitute $x = u^2$ to get

$$\int u\ln(u^2)\,2u\,du = 4\int u^2\ln(u)\,du.$$

IBP to get

$$
\begin{aligned}
&= \tfrac{4}{3}u^3 \ln(u) - \tfrac{4}{3}\int u^3\, u^{-1}\, du \\
&= \tfrac{4}{3}u^3 \ln(u) - \tfrac{4}{9}u^3 + C \\
&= \tfrac{4}{3}x^{3/2} \ln(x^{1/2}) - \tfrac{4}{9}x^{3/2} + C \\
&= \tfrac{2}{3}x^{3/2} \ln(x) - \tfrac{4}{9}x^{3/2} + C.
\end{aligned}
$$

□

(1c) $\int x^{-3}(x^2-1)^{-1/2}\, dx$

Solution. Substitute $x = \sec(\theta)$ with $\theta = \operatorname{arcsec}(x)$. Then $dx = \sec(\theta)\tan(\theta)\, d\theta$ and we obtain

$$
\begin{aligned}
&= \int \sec(\theta)^{-3}(\sec^2(\theta)-1)^{-1/2}\ \sec(\theta)\tan(\theta)\, d\theta \\
&= \int \sec(\theta)^{-3}\tan(\theta)^{-1}\ \sec(\theta)\tan(\theta)\, d\theta \\
&= \int \sec(\theta)^{-2}\, d\theta \\
&= \int \cos^2(\theta)\, d\theta \\
&= \int \tfrac{1}{2}(1+\cos(2\theta))\, d\theta \\
&= \tfrac{1}{2}\theta + \tfrac{1}{4}\sin(2\theta)\, d\theta \\
&= \tfrac{1}{2}\theta + \tfrac{1}{2}\sin(\theta)\cos(\theta)\, d\theta.
\end{aligned}
$$

Now $\cos(\theta) = x^{-1}$ while $\sin(\theta)$ has the same sign as θ and satisfies $\sin^2(\theta) = 1 - \cos^2(\theta) = 1 - x^{-2} = \frac{x^2-1}{x^2}$ so that

$$\sin(\theta) = x^{-1}\sqrt{x^2-1}.$$

The final answer is

$$\tfrac{1}{2}\operatorname{arcsec}(x) + \tfrac{1}{2}x^{-2}(x^2-1)^{1/2} + C.$$

□

(2a) Indicate the form of the partial fractions decomposition of $\frac{x^2-3}{x(x^2-1)(x^2+4)^2}$.

Solution.

$$\frac{a_1}{x} + \frac{a_2}{x-1} + \frac{a_3}{x+1} + \frac{b_1x+c_1}{(x^2+4)^2} + \frac{b_2x+c_2}{(x^2+4)}.$$

□

(2b) Find the partial fractions decomposition of $\frac{x^2-3}{x^2(x^2+6)}$. Show your work and reasoning.

Solution. Write
$$\frac{x^2-3}{x^2(x^2+6)} = \frac{a}{x^2} + \frac{b}{x} + \frac{cx+d}{x^2+6}.$$
Cross multiply to get
$$x^2 - 3 = a(x^2+6) + bx(x^2+6) + (cx+d)x^2 = (b+c)x^3 + (a+d)x^2 + 6bx = 6a.$$
Therefore
$$a = -\tfrac{1}{2}, \qquad b = 0, \qquad b+c = 0, \qquad a+d = 1.$$
Therefore
$$b = 0 \text{ and } d = \tfrac{3}{2}.$$
Final answer:
$$-\frac{1}{2}\cdot\frac{1}{x^2} + \frac{3}{2}\cdot\frac{1}{x^2+6}.$$
□

(3a) Find the area under the graph $y = x^{-1/3}$ from $x = 0$ to $x = 4$. Show all steps of your reasoning.

Solution. This is an improper integral, with integrand unbounded as $x \to 0^+$.
$$\int_t^4 x^{-1/3}\,dx = \tfrac{3}{2}x^{2/3}\Big]_t^4 = \tfrac{3}{2}(4^{2/3} - t^{2/3}).$$
Now
$$\lim_{t\to 0^+}\int_t^4 x^{-1/3}\,dx = \frac{3}{2}\cdot 4^{2/3}.$$
□

(3b) Determine whether the improper integral $\int_{10}^{\infty} e^{-1/x}\,(1+x^6)^{-1}\,dx$ converges.

Solution. Let f be the given integrand, and define $g(x) = x^{-6}$. Then $f(x) \le g(x)$ for all $x \ge 10$ since
$$\frac{e^{-1/x}}{1+x^6} \le \frac{1}{1+x^6} \le \frac{1}{x^6} = x^{-6}.$$
We learned that improper integrals $\int_a^{\infty} x^{-p}\,dx$ converge for any lower limit of integration $a > 0$ if $p > 1$. Here $p = 6$, so $\int_{10}^{\infty} g(x)\,dx$ converges. By the comparison theorem, the given integral also converges. □

(4) Let $f(x) = x^2 \cos(x)$. How large should we take n in order to guarantee that the Midpoint Rule approximation M_n to $\int_{2\pi}^{4\pi} f(x)\,dx$ is accurate to within 10^{-8}? (That is, $|\int_{2\pi}^{4\pi} f(x)\,dx - M_n| < 10^{-8}$.)

Solution. We use the midpoint rule error bound $|E_{M_n}| \le \frac{(b-a)^3 K}{24n^2}$. Here $b - a = 2\pi$. To find an OK value for K we use

$$\begin{aligned}|f''(x)| &= \big| - x^2 \cos(x) - 4x \sin(x) + 2\cos(x)\big| \\ &\le x^2|\cos(x)| + 4x|\sin(x)| + 2|\cos(x)| \\ &\le x^2 + 4x + 2 \\ &\le 4\pi)^2 + 16\pi + 2.\end{aligned}$$

Define

$$K = 4\pi^2 + 16\pi + 2.$$

Now find n satisfying $\frac{(b-a)^3 K}{24n^2} < 10^{-8}$. This is equivalent to

$$n^2 > 10^8(2\pi)^3 K/24.$$

Thus we choose n to be the smallest positive integer that satisfies

$$n > 10^4(2\pi)^{3/2}(4\pi^2 + 16\pi + 2)^{1/2}/\sqrt{24}.$$

□

(5) How does the Mean Value Theorem enter into the derivation of the integral formula for arclength?

Solution. The length of the curve is the limit, as $\Delta x \to 0$, of the sum

$$\Delta x \sum_{i=1}^{n} \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{\Delta x}\right)^2}$$

of the lengths of approximating line segments.
According to the Mean Value Theorem,

$$\frac{f(x_i) - f(x_{i-1})}{\Delta x} = f'(x_i^*)$$

for some unknown points x_i^* in $[x_{i-1}, x_i]$. Thus the sum is expressed as a Riemann sum approximating $\int_a^b \sqrt{1 + (f'(x))^2}\,dx$. □