## Q1 True or False
20 Points

Select True (i.e., always true) or False (i.e., sometimes false) for each statement. No need to provide an explanation.

### Q1.1 Nullspace
2 Points

If $W$ is a $3-$dimensional subspace of $\mathbb{R}^5$ then there is a $2\times 5$ real matrix $A$ such that $W$ is the nullspace of $A$.

### Q1.2 Eigenspace
2 Points

If $W$ is a $3-$dimensional subspace of $\mathbb{R}^5$ then there is a $5\times 5$ real matrix $A$ such that $W$ is the eigenspace of $A$ corresponding to $\lambda = 1$.

### Q1.3 Subspace
2 Points

Let $M_2$ be the vector space of $2\times 2$ real matrices with entrywise addition and scalar multiplication. Then the set

$$H = \{X \in M_2 : X^2 = 0\}$$

is a subspace of $M_2$.

### Q1.4 Composition
2 Points

If $T : V \to V$ and $S : V \to V$ are invertible linear transformations, then

$$S \circ T \circ S^{-1} : V \to V$$

### Q1.5 Distinct
2 Points

If $A$ is a diagonalizable real $n\times n$ matrix, then it must have $n$ distinct eigenvalues.

### Q1.6 Invertible
2 Points

If $A$ is an $m\times n$ matrix and $A^T A$ is invertible then $AA^T$ is invertible.

### Q1.7 Similar Rank
2 Points

If $A$ is similar to $B$ then $\operatorname{rank}(A) = \operatorname{rank}(B)$.

### Q1.8 Row Ops
2 Points

If $A, B$ are $m\times n$ matrices and $A$ is row equivalent to $B$ then every vector $b \in \mathbb{R}^m$ has the same distance from $\operatorname{Col}(A)$ and $\operatorname{Col}(B)$.

### Q1.9 Zero
2 Points

If $x, y$ are linearly independent vectors in $\mathbb{R}^2$ and $z \in \mathbb{R}^2$ satisfies

$$z\cdot x = z\cdot y = 0$$

then $z = 0$.

### Q1.10 Coordinate Dot Product
2 Points

If $x, y \in \mathbb{R}^n$ are orthogonal and $\mathcal{B}$ is a basis of $\mathbb{R}^n$, then $[x]_{\mathcal{B}} \cdot [y]_{\mathcal{B}} = 0$.

## Q2 Honor Code + Cheat Sheet + Instructions
10 Points

## Q3 Examples
20 Points

Give an example of each of the following, explaining why it has the required property, or explain why no such example exists.

### Q3.1 Isomorphic
5 Points

A $2\times 2$ real matrix whose null space is isomorphic to its column space.

### Q3.2 Diagonalizable
5 Points

### Q3.3 Coordinates
5 Points

An orthonormal basis $\mathcal{B} = \{b_1, b_2\}$ of $\mathbb{R}^2$ such that $[b_1]_{\mathcal{B}} = b_2$.

### Q3.4 Projection
5 Points

A subspace $W$ of $\mathbb{R}^3$ such that $W \neq \mathbb{R}^3$ and the orthogonal projection

$$\operatorname{proj}_W : \mathbb{R}^3 \to \mathbb{R}^3$$

is one to one.

## Q4 Polynomials
11 Points

Let $P_2 = \{a_0 + a_1 t + a_2 t^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ denote the vector space of polynomials of degree at most two with real coefficients. Consider the linear transformation $T : P_2 \to \mathbb{R}^3$ by

$$T(p) = \begin{bmatrix} p(0) \\ p'(0) \\ p''(0) \end{bmatrix},$$

where $p', p''$ denote first and second derivatives. Show that $T$ is invertible and find its inverse. Explain your reasoning.

## Q5 Transpose
13 Points

Let

$$M_2 = \left\{ \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} : x_{ij} \in \mathbb{R} \right\}$$

be the vector space of $2\times 2$ real matrices with entrywise addition and scalar multiplication. Consider the linear transformation $S : M_2 \to M_2$ by

$$S(X) = X - X^T.$$

(a) (10pts) Find a basis $\mathcal{B}$ of $M_2$ in which ${}_{\mathcal{B}}[S]_{\mathcal{B}}$ (i.e., the matrix of $S$ with respect to the $\mathcal{B}$ basis) is diagonal, and write that diagonal matrix. Explain your reasoning.

(b) (3pts) Find a basis for the kernel of $S$.

## Q6 High Power
13 Points

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) (10pts) What is the second column of $B = A^{99}$? Explain your reasoning.

(b) (3pts) Show that $A$ is invertible. What is the second column of $C = A^{99} - A^{-99}$? (Here $A^{-99}$ means $(A^{-1})^{99}$).

## Q7 Distance to Nullspace
13 Points

Let

$$A = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(a) (10pts) Find the vector $\hat{b} \in \operatorname{Nul}(A)$ which is closest to $b$. Explain your reasoning.

(b) (3pts) Letting $W = \operatorname{Nul}(A)$, find vectors $y \in W$ and $z \in W^{\perp}$ such that

$$b = y + z.$$