Midterm 2

Name:

SID:

Rules and Guidelines

- **This checkpoint is out of 101 points and you will have a week to complete it.**
- Answer all questions. Read them carefully first. Not all parts of a problem are weighted equally.
- Be precise and concise.
- The problems may **not** necessarily follow the order of increasing difficulty.
- Any algorithm covered in the lecture can be used as a blackbox unless stated otherwise.
- Please either latex your answers, or provide a neatly handwritten version of the solutions where every subpart is on a new page of your submission and the number is clearly labelled.
- Good luck!

Discussion Section

Which of these do you consider to be your primary discussion section(s)? Feel free to choose multiple, or to select the last option if you do not attend a section.

(a) Fred Zhang Wheeler 202 • Tu 5-6pm

(b) Jialin Li Etcheverry 3105 • Tu 5-6pm

(c) Tiffany Chien Etcheverry 3109 • W 9-10am

(d) Sidhanth Mohanty Wheeler 130 • W 9-10am

(e) Teddy Tran Moffitt Library 106 • W 10-11am

(f) Emaan Hariri Hearst Field Annex B5 • W 12-1pm

(g) Dee Guo Wheeler 224 • W 12-1pm

(h) Arpita Singhal Dwinelle 234 • W 1-2pm

(i) Gillian Chu Barrows 136 • W 1-2pm

(j) Rachel De Jaen Barrows 155 • W 1-2pm

(k) Joshua Turcotti Wheeler 20 • W 2-3pm

(l) Varun Jhunjhunwalla Wheeler 202 • W 2-3pm

(m) Vishnu Iyer Etcheverry 3113 (Advanced Section) • W 2-3pm

(n) Avni Singhal Wheeler 30 • W 3-4pm

(o) Jiazheng Zhao Dwinelle 229 • W 3-4pm

(p) Rishi Veerapaneni Wheeler 224 • W 3-4pm

(q) Jeff Xu Evans 9 • W 3-4pm

(r) Noah Kingdon Wheeler 202 • W 5-6pm

(s) Neha Kunjal Hildebrand B51 • Th 11-12pm

(t) Christina Jin Evans 9 • Th 12-1pm

(u) Noah Krakoff Moffitt Library 103 • Th 1-2pm

(v) Ajay Raj Wheeler 202 • Th 2-3pm

(w) Don't attend section.

1 True or False? (14 points)

For each part, give a brief proof if you think it's true, or provide a counter example if you believe it is false.

(a) (**2 points**) For any undirected graph with all positive edge weights, there exists a vertex v such that the tree of shortest paths rooted at v is an MST of the graph.

◯ True ◯ False

(b) (**2 points**) Consider a linear program with a minimization objective. If there are infinitely many integral feasible solutions, where all variables take integral values, then the objective value can be made arbitrarily small.

◯ True ◯ False

(c) (**2 points**) Let A be a $m \times n$ matrix, b be a m-dimensional vector, and c be a n-dimensional vector. If the linear program

$$\min_x c^\top x \quad \text{such that} \quad Ax \geq b, x \geq 0$$

has a feasible solution with objective value M, then every feasible point of its dual has an objective value at most M.

◯ True ◯ False

(d) (**2 points**) If a graph has a unique lightest edge, then that edge must be part of *every* MST of the graph.

◯ True ◯ False

(e) (**2 points**) Let F_1 and F_2 be feasible s-t flows in a directed graph G, satisfying the flow conservation and capacity constraints. Then the flow F defined as $F(e) := 0.6F_1(e) + 0.4F_2(e)$ for all $e \in E$ is also a feasible s-t flow in G.

◯ True ◯ False

(f) (**2 points**) We say an edge is *saturated* by a flow if the amount of flow on the edge equals its edge capacity. In a directed graph, let e be an edge saturated by some maximum s-t flow of value F. Let c be its capacity. If we reverse e, then in the new graph any maximum s-t flow must have value at most $F - c$.

◯ True ◯ False

(g) (**2 points**) Consider maximum s-t flow in a directed graph. Let T be the number of incoming edges of t. If we add a constant $C > 0$ to all edge capacities, then the value of maximum s-t flow will increase by $C \cdot T$.

◯ True ◯ False

2 Quick Union-Find (6 points)

Consider a disjoint set data structure $\mathcal{S}$ instantiated on the universe $\{1, \ldots, n\}$. Some arbitrary number of Union and Find operations has been performed on $\mathcal{S}$.

(a) (**2 points**) Assume $\mathcal{S}$ only uses union-by-rank, without path compression. Give the tightest possible upper bound on the average time complexity of running 1000 additional Find operations. Give your answer in big-O notation as a function of n.

(b) (**2 points**) Assume $\mathcal{S}$ uses union-by-rank **with path compression**. Give the tightest possible upper bound on the average time complexity of running 1000 additional Find operations. Give your answer in big-O notation as a function of n.

(c) (**2 points**) Assume $\mathcal{S}$ uses union-by-rank **with path compression**. Then the time complexity of each individual Find is at most $\mathcal{O}(\log^* n)$. Mark if this is true or false, and justify.

◯ True ◯ False

3 Space/Time complexity (9 points)

Define a function $S(i, j)$ recursively by:

$$S(i,j) = \max\left\{S(i-2, j-1) + 4, S(i-1, j-1) + 7, S(i-1, j)\right\}$$

when $i \in [0, m]$, $j \in [0, n]$ and $m < n$.

Notice how this function is not fully specified due to a lack of base cases.

(a) (**3 points**) Describe a minimal set of base cases that would make this a well-defined recurrence. We call (i, j) a base case if it is not in $[0, m] \times [0, n]$. Set the value of S for each base case to be 0. Make sure to only include the base cases that are necessary.

(b) (**2 points**) For all possible algorithms, give the tightest asymptotic bound on the **space complexity** of computing $S(m, n)$ using dynamic programming/memoization.

(c) (**4 points**) Which of the following recurrence relations would take $O(n^2)$ time to compute if memoized and all subproblems in the specified range have to be computed? You may assume that each of the functions below are equal to 0 if any of their inputs are negative.
Note: there may be multiple answers.

i) $S(1) = 5, S(x) = \min_{1 \leq k \leq x-1} S(k) + S(k)^2$ for $x \in [2, n]$.

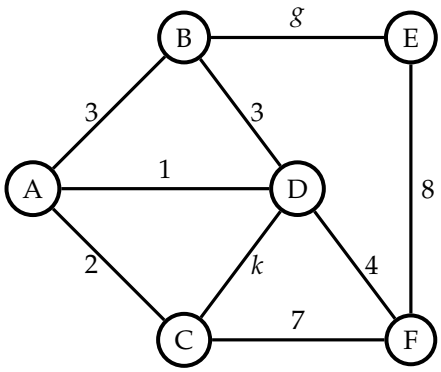
ii) $S(i, j) = \max\{S(i-3, j) + 9, S(i, j-1) + 3\}$ where $i \in [0, n], j \in [0, n]$

iii) $S(i, j, k) = \max\{S(i-3, j, k-1) + 2, S(i, j-1, k-1) + 3\}$ where $i \in [0, n], j \in [0, n], k \in [0, n/2]$

iv) $S(i, j, k) = \max\{S(i-3, j, k) + 9, S(i, j-1, k-1) + 3\}$ where $i \in [0, n], j \in [0, n], k \in [0, 3]$

4 To choose or not to choose (12 points)

We are given the following graph G:



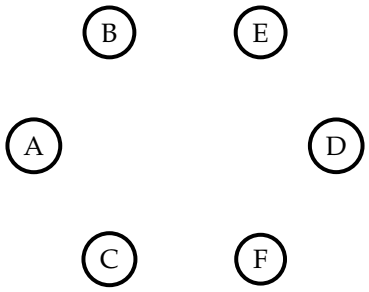
a) (**2 points**) For what range of values of g are you **guaranteed** to include edge BE in the MST?

b) (**2 points**) For what range of values of k are you **guaranteed** to include edge CD in the MST?

c) (**2 points**) For what range of values of k are you **guaranteed to NOT** include edge CD in the MST?

d) (**3 points**) Suppose an adversary can set g and k arbitrarily. What is the maximum cost of a MST that the adversary can force?

e) (**3 points**) Draw seven edges on the following graph with six vertices, and then mark six edges with 1 and one edge with x such that an adversary can choose x such that the cost of the MST is arbitrarily large.



5 Find the unknown coefficient (9 points)

(a) (**4 points**) Consider the following linear program in which the value of k is unknown. Find the range of values of k such that the point $(x_1, x_2) = (3, 5)$ is optimal.

$$\begin{aligned} \max\, & 2x_1 + x_2 \\ x_1 + x_2 &\geq 2 \\ x_1 &\leq 3 \\ kx_1 + x_2 &\leq 3k + 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(b) (**3 points**) Construct the dual of the linear program in part a. Leave k in your answer.

(c) (**2 points**) Convert the linear program **in part a** to be of the form where your objective function is a minimization function, and your constraints are all in the form variable $\geq$ equation (e.g. $x_1 \geq 5$). Leave k in your answer.
Note that this question is not asking you to find the dual

6 Trees of Candles (14 points)

You are an interior decorator, confronted with a dark living room. To lighten the room up, you have n candles and want to build k trees of candles. First, the candles $\{1, \ldots, k\}$ are already put on the floor, to serve as roots of the k trees. You would like to put the remaining $n - k$ candles $\{k+1, \ldots, n\}$ on top of them to form k trees $\{T_1, \cdots, T_k\}$ of candles, where each T_i is rooted at candle i.

You can put an arbitrary number of candles on top of the same candle. The cost to putting candle i on top of candle j is $C(i, j)$. We assume that $C(i, j) \geq 0$ and $C(i, j) = C(j, i))$ for $i \neq j$. No candle can be reused or put on top of itself. You cannot use the candles $\{1, \ldots, k\}$ anymore, as they have already been fixed to the floor.

(a) (**6 points**) Describe an algorithm to find a solution of minimum cost.

(b) (**5 points**) Prove the correctness of your algorithm.

(c) (**3 points**) Give a runtime analysis of your algorithm in the previous part.

7 Fancy Sequence (11 points)

Let $A[1], \ldots, A[n]$ be an array. Call a (not necessarily contiguous) subsequence $[A[i_1], \ldots, A[i_k]]$ *fancy* if $k \leq 2$, or if for all $j \notin \{1, k\}$, we have $17A[i_j] < A[i_{j+1}] + A[i_{j-1}]$.

This question asks you to provide an algorithm to compute the length of the longest fancy subsequence of a given length-n array A. The sequence need not be contiguous. You may assume arithmetic operations (addition, division and multiplication) and comparisons ($<$, $>$, $==$) can be done in constant time.

(a) (**6 points**) For $i < j$, let $f(i, j)$ denote the length of the longest fancy subsequence of $A[i..n]$ whose first two elements are $A[i]$ and $A[j]$. Write a recurrence relation for $f(i, j)$ and specify the base case(s).

(b) (**3 points**) Give a polynomial-time algorithm to compute the length of the longest fancy subsequence of A. You may write pseudocode.

(c) (**2 points**) Give a runtime analysis of your algorithm.

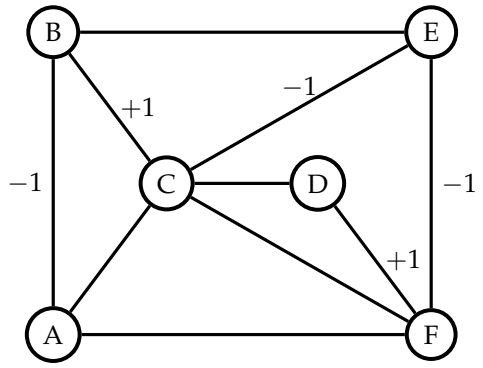
8 Switchings and Signings (14 points)

Let $G = (V, E)$ be a connected, undirected graph on n vertices as input. Every edge e has a sign $\sigma(e) \in \{-1, +1\}$, initially equal to $+1$. We will call an assignment of signs to all the edges in G a *signing*.

We call a signing σ *balanced* if **for every cycle** C in G, the product of signs of edges in the cycle is equal to 1, i.e.,

$$\prod_{e \in C} \sigma(e) = 1.$$

(a) (**2 points**) A partial signing is given for the below graph. Sign the remaining edges so that the resulting signing is balanced.



BE	
CD	
CF	
AC	
AF	

You are allowed to perform the following operation known as a *switching* — choose a vertex v and toggle the sign of all edges incident to v, i.e., update the sign on every edge e incident to v from $\sigma(e)$ to $-\sigma(e)$.

(b) (**4 points**) Prove that any signing of G constructed via a sequence of *switching* operations must be balanced. *Recall that initially all the edge signs are* $+1$.

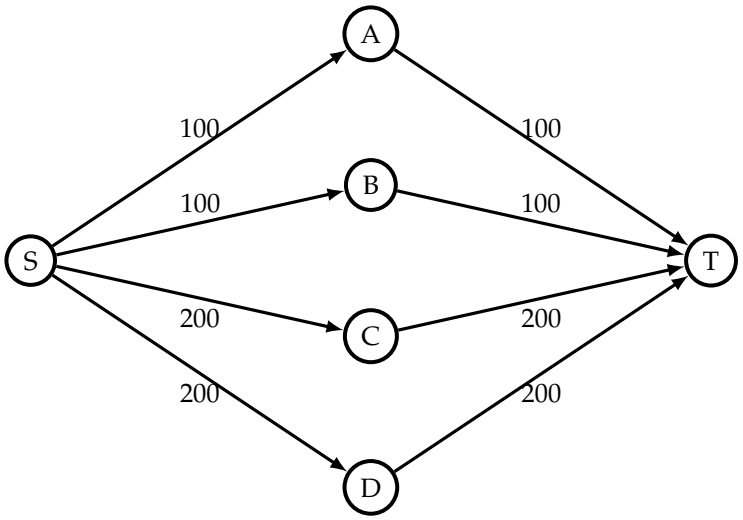
(c) (**4 points**) Let T be a spanning tree of G, and σ_T be an arbitrary signing of edges in T. Prove that there is a unique balanced signing σ of G such that $\sigma(e) = \sigma_T(e)$ for all $e \in T$.
Hint: first try to show that there is at least one such balanced signing (part (b) might be useful for this), and next show that there is at most one balanced signing that agrees with σ_T.

(d) (**4 points**) Let σ be a balanced signing of G. Show that there exists a sequence of *switching* operations that produces σ and give an algorithm that takes in G and σ as input and produces this sequence of *switching* operations. Prove the correctness and analyze the running time of your algorithm.
Hint: consider using the statement in part (c).

9 Flowww (12 points)

Recall that in each iteration of the Ford-Fulkerson method, we push as much flow as possible along an augmenting path from S to T. For this problem, consider all possible executions of Ford-Fulkerson, which can use any augmenting path, and Edmonds-Karp, an implementation of Ford-Fulkerson that uses BFS to select the augmenting path in each iteration. All edge capacities in the network must be non-negative integers.

a) (**6 points**) Add one directed edge and its capacity to the graph below that **maximizes** the **largest possible** number of iterations of Ford-Fulkerson. If no edge can be added to strictly increase the maximum number of iterations, leave the graph blank and fill in the "No such edge" box.



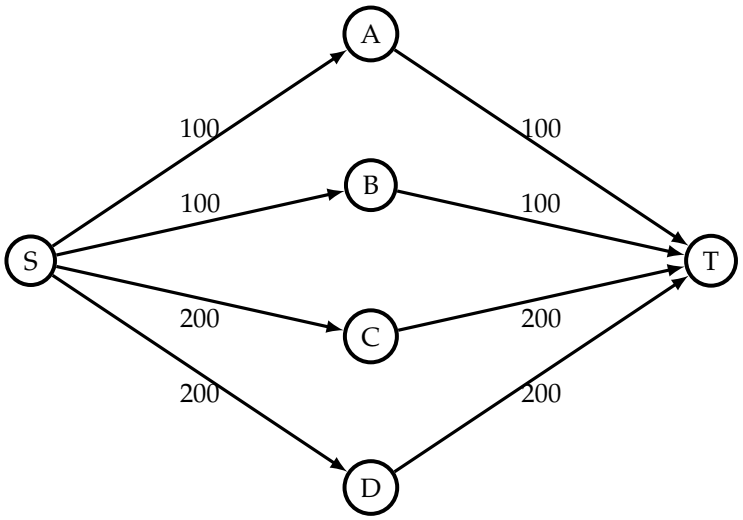
☐ No such edge

i) What is the new maximum number of iterations? ____

ii) What are the paths of the flow, as well as the units pushed across each path, found by the algorithm once it is run to completion? You may not need all the rows in the table.

Path	Units

b) (**6 points**) Add one directed edge and its capacity to the graph below that **maximizes** the **largest possible** number of iterations of Edmonds-Karp. If no edge can be added to strictly increase the maximum number of iterations, leave the graph blank and fill in the "No such edge" box.



☐ No such edge

i) What is the new maximum number of iterations?

ii) What are the paths of the flow, as well as the units pushed across each path, found by the algorithm once it is run to completion? You may not need all the rows in the table.

Path	Units