Note: some problems are worded differently here than they were on your exam.

MATH 54 MIDTERM 2 – November 14, 2019, 5:10-6:30pm

Your Name Student ID	SOLUTIONS

Please exchange student IDs to record the

names of your two closest seat neighbors	

Do not turn this page until you are instructed to do so.

Show all your work in this exam booklet. There are blank pages for scratch work, but please do not remove any pages! **If you want something on an extra page to be graded, label it by the problem number and write "XTRA" on the page of the actual problem.** *In the event of an emergency or fire alarm leave your exam on your seat and meet with your GSI or professor outside.*

This exam consists of 4 problems, each of which has parts (a) and (b), in the general topic areas 1) dimension and coordinate systems, 2) eigenvalues and eigenvectors, 3) second order ODEs, 4) first order linear ODE systems.

Point values are indicated in brackets to the left of each problem, add up to a total of 80, and so can be used as guide for managing the 80 minute exam time.

Each part of (a) yields full or no credit, and you don't need to show work. **To ensure credit please put each answer (and only the final answer) into the given box. Empty boxes will receive automatic score 0, so if your answer is elsewhere, put at least an arrow into the box.**

Parts (b) can yield partial credit, in particular for explanations and documentation of your approach, even when you don't complete a calculation. In particular, if you recognize your result to be wrong (e.g. by checking!), stating this will yield partial credit. On the other hand, wrong or irrelevant statements mixed with correct work may result in reduced credit.

When asked to explain/show/prove, you should make clear and unambiguous statements that would be accessible to another student. In particular, use words or arrows to indicate how formulas relate to each other. *You may use any theorems or facts stated in the lecture notes, script, and the book sections covered by the course up to Nov.8 – after stating them clearly. If you use theorems or facts that you know from other sources, you will obtain full credit only if you include proofs that derive them from the current course material.*

(5)

[3] **1a)** The dimension of a subspace $H \subset \mathbb{R}^n$ is defined to be ...

the number of vectors in a basis of H

[2] Given the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^2$, the vector $\mathbf{x} \in \mathbb{R}^2$ with coordinates $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is

$$3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+8 \\ 6+4 \end{bmatrix}$$

$\mathbf{x} = \boxed{\begin{bmatrix} 11 \\ 10 \end{bmatrix}}$.

[3] Given the basis $\mathcal{B} = \{2t^2 - 1, 4t - 3, 6t^2\}$ and the basis $\mathcal{C} = \{2t^2, 2t, 1\}$ of $\mathbb{P}_2$, compute the change-of-coordinates matrix defined by $P[x]_{\mathcal{B}} = [x]_{\mathcal{C}}$.

$$P = \left[[b_1]_C \; \dots \; [b_3]_C \right]$$

$$\begin{matrix} \| \\ 1 \cdot 2t^2 \\ + 0 \cdot 2t \\ + (-1) \cdot 1 \end{matrix} \qquad \begin{matrix} 4t-3 \\ \| \\ 0 \cdot 2t^2 \\ + 2 \cdot 2t \\ + (-3) \cdot 1 \end{matrix} \qquad \begin{matrix} 6t^2 \\ \| \\ 3 \cdot 2t^2 \\ + 0 \cdot 2t \\ + 0 \cdot 1 \end{matrix}$$

$P = \boxed{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ -1 & -3 & 0 \end{bmatrix}}$.

[2] Given vectors $v_1, \dots, v_k$ in a vector space V and a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ of V, write "none", "$\Rightarrow$", "$\Leftarrow$", or "$\Leftrightarrow$" into the box for the implications between the following statements:

$v_1, \dots v_k$ are linearly independent in V $\boxed{\Leftrightarrow}$ $[v_1]_{\mathcal{B}}, \dots [v_k]_{\mathcal{B}}$ are linearly independent in $\mathbb{R}^n$

"$\Rightarrow$" because coordinate mapping is linear

"$\Leftarrow$" ——— linear & one-to-one

[10] **1b)** The rank theorem can be formulated for any linear transformation $T : V \to W$ between finite dimensional vector spaces V, W. It says that $\dim \operatorname{range}(T) + \dim \operatorname{kernel}(T) = \dim V$.
Use this to explain ~~the following two facts~~: one of the following facts:

(i) A homogeneous system of m linear equations for n variables has at least $n - m$ free variables.

(ii) A linear transformation $T : V \to V$ (for V finite dimensional) is onto if and only if it is one-to-one.

(i) Solution set of the system is kernel of a matrix transformation $T: \mathbb{R}^n \to \mathbb{R}^m$, so

$$\#\text{free variables} = \dim \operatorname{kernel} T = \underbrace{\dim \mathbb{R}^n}_{=n} - \underbrace{\dim \operatorname{range}(T)}_{\leq m}$$

$$\geq n-m$$

range(T) is a subspace of $\mathbb{R}^m$, so dimension is $\leq m$

(ii) T one-to-one $\Leftrightarrow$ kernel$(T) = \{0\}$ $\Leftrightarrow$ $\dim \operatorname{kernel}(T) = 0$

(any other subspace has a basis so $\dim \geq 1$)

T onto $\Leftrightarrow$ range$(T) = V$ $\Leftrightarrow$ $\dim \operatorname{range}(T) = \dim V$

(If $H \subset V$ has basis with $\dim V$ vectors, then these $\dim V$ linearly independent vectors also span V (by a Thm))

$\overset{\text{rank thm}}{\Longleftrightarrow}$ $\dim \operatorname{range}(T) = \dim \operatorname{range}(T) + \dim \operatorname{kernel}(T)$

$\Leftrightarrow$ $0 = \dim \operatorname{kernel}(T)$

$\underset{\text{see above}}{\Longleftrightarrow}$ T one-to-one

[3] **2a**) The eigenvalues of $\begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix}$ are $\boxed{4+i\,,\ 4-i}$.

$0 = (5-\lambda)(3-\lambda)+2$

$= \lambda^2 - 8\lambda + 15 + 2 \quad \rightsquigarrow \lambda = 4 \pm \sqrt{16-17}$

[3] A basis for the $\lambda = 4$ eigenspace of $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ is $\boxed{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$.

$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 + x_2 + x_3 = 0$, x_2, x_3 free

[4] If a matrix satisfies $A \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \sqrt{2} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $A \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \sqrt{2} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $A \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = -5 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

then we can diagonalize it $A = PDP^{-1}$ with

$P = \boxed{\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 3 & 0 & 1 \end{bmatrix}}$ and $D = \boxed{\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -5 \end{bmatrix}}$.

[10] **2b)** Define the notion of similarity between matrices A and B.

Then, given $A = \begin{bmatrix} -10 & 50 \\ -5 & 20 \end{bmatrix}$, find a similar matrix of the form $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ by specifying the similarity transformation as well as $a, b \in \mathbb{R}$.

A is similar to B if $P^{-1}AP = B$ for some invertible P

<u>complex eigenvalue?</u>

$$\det(A - \lambda I) = (-10-\lambda)(20-\lambda) + 5 \cdot 50 = \lambda^2 + 10\lambda - 20\lambda - 200 + 250$$
$$= \lambda^2 - 10\lambda + 50$$

roots $\lambda = 5 \pm \sqrt{25 - 50} = 5 \pm 5i$

<u>$5-5i$ eigenvector</u>

$$\begin{bmatrix} -10-5+5i & 50 \\ -5 & 20-5+5i \end{bmatrix} \sim \begin{bmatrix} -3+i & \overset{(3+i)(3-i)}{\overset{\shortparallel}{10}} \\ -1 & 3+i \end{bmatrix} \sim \begin{bmatrix} 1 & -3-i \\ 0 & 0 \end{bmatrix}$$

$x_1 - (3+i)x_2 = 0$

x_2 free

$$\underline{v} = \begin{bmatrix} 3+i \\ 1 \end{bmatrix} \xleftarrow{x_2 = 1}$$

By a Theorem,

$$P^{-1}AP = \underbrace{\begin{bmatrix} 5 & -5 \\ 5 & 5 \end{bmatrix}}_{\text{rotation matrix with } r = \sqrt{5^2+5^2} = 5\sqrt{2}} \quad \text{with } P = [\operatorname{Re} v \ \operatorname{Im} v] = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\cos\theta = \frac{5}{5\sqrt{2}} = \frac{\sqrt{2}}{2} = \sin\theta \rightsquigarrow \theta = \pi/4$$

<u>Alternative</u>: using $5+5i$ eigenvector yields

$$P^{-1}AP = \begin{bmatrix} 5 & 5 \\ -5 & 5 \end{bmatrix} \text{ with } P = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } r = 5\sqrt{2}, \theta = -\pi/4$$

[3] **3a)** The general solution of $y''(t)+4\,y(t)=0$ is $\boxed{y(t)=C_1\cos 2t+C_2\sin 2t \quad \text{with } C_1,C_2\in\mathbb{R}}$.

$r^2+4=0$

$r=\pm 2i \rightsquigarrow$ complex solution $e^{2it}=\cos 2t+i\sin 2t$

$\rightsquigarrow$ real & imaginary part

[2] A solution of $y''(t)+y(t)=t^2-1$ can be found using the Ansatz
(Hint: Give a formula for y with unknown coefficients. You do not need to compute the coefficients.)

$\boxed{y(t)=A+Bt+Ct^2 \quad A,B,C\in\mathbb{R} \text{ unknown}}$.

[3] The general solution of $y''(t)+2y'(t)+y(t)=0$ is $\boxed{y(t)=C_1e^{-t}+C_2te^{-t} \quad \text{with } C_1,C_2\in\mathbb{R}}$.

$r^2+2r+1=0$

$r=-1\pm\sqrt{1-1}=-1$ multiplicity 2

[2] A solution of $y''(t)+2y'(t)+y(t)=e^{3t}\cos t$ can be found with the Ansatz
(Hint: Give a formula for y with unknown coefficients. You do not need to compute the coefficients.)

$\boxed{y(t)=Ae^{3t}\cos t+Be^{3t}\sin t \quad A,B\in\mathbb{R} \text{ unknown}}$.

[10] **3b)** Determine the general solution y of $a(t)y''(t) + b(t)y'(t) = t^3$ from the following information, and prove your formula without the use of theorems.

(1) $a, b \in \mathcal{C}^\infty$ are given so that $T : \mathcal{C}^\infty \to \mathcal{C}^\infty,\ y \mapsto ay'' + by'$ is a linear transformation.

(2) $\cancel{y}(t) = t^4$ satisfies $a(t)\cancel{y}''(t) + b(t)\cancel{y}'(t) = 8t^3$.

(3) The general solution of $a(t)y''(t) + b(t)y'(t) = 0$ is $y(t) = c_1 + c_2 t^2$ for $c_1, c_2 \in \mathbb{R}$.

general solution = particular solution + general homogeneous solution

$$y(t) = \tfrac{1}{8}t^4 + c_1 + c_2 t^2 \quad \text{with } c_1, c_2 \in \mathbb{R}$$

<u>Proof</u>: $T[\tfrac{1}{8}t^4 + c_1 + c_2 t^2] = \tfrac{1}{8}\underbrace{T[t^4]}_{\substack{=8t^3 \\ \text{by (2)}}} + \underbrace{T[c_1 + c_2 t^2]}_{=0 \text{ by (3)}} = t^3$

To see that there are no further solutions, assume $y \in \mathcal{C}^\infty$ solves $T[y] = t^3$, then

$$T[y - \tfrac{1}{8}t^4] = T[y] - \tfrac{1}{8}T[t^4] = t^3 - \tfrac{1}{8}\cdot 8t^3 = 0$$

So $y - \tfrac{1}{8}t^4$ solves the homogeneous equation, and by (2)

$$y(t) - \tfrac{1}{8}t^4 = c_1 + c_2 t^2 \text{ for some } c_1, c_2 \in \mathbb{R}$$

This implies that $y(t) = \tfrac{1}{8}t^4 + c_1 + c_2 t^2$ is of the claimed form.

$x_3' = y''' = 5y'' + 0y' - 3y + e^t = 5x_3 + 0x_2 - 3x_1 + e^t$

[4] **4a)** The differential equation $y'''(t) - 5\,y''(t) + 3y(t) = e^{t^2}$ is equivalent to a first order system

$$\boxed{\underline{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 0 & 5 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ e^{t^2} \end{bmatrix}}$$

by the substitution $\boxed{\underline{x} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}}$.

[3] If a matrix satisfies $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \sqrt{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $A \begin{bmatrix} 2 \\ 3 \end{bmatrix} = -10 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ then the general solution of

the ODE system $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x}(t) = \boxed{c_1 e^{\sqrt{2}t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{-10t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ with } c_1, c_2 \in \mathbb{R}}$.

[3] If a real 2×2 matrix A satisfies $A \begin{bmatrix} i \\ 1 \end{bmatrix} = (3 + 2i) \begin{bmatrix} i \\ 1 \end{bmatrix}$ then the general solution of

the ODE system $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x}(t) = \boxed{c_1 e^{-3t} \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} \text{ with } c_1, c_2 \in \mathbb{R}}$.

complex solution

$$e^{(3+2i)t} \begin{bmatrix} i \\ 1 \end{bmatrix} = e^{-3t}(\cos 2t + i \sin 2t)\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$= e^{-3t} \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} + i\, e^{-3t} \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}$$

reals imaginary parts

[10] **4b)** Find the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = \begin{bmatrix} 0 & 2 \\ 4 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -2 \\ -6 \end{bmatrix}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

<u>homogeneous equation</u>

eigenvalues : roots of $-\lambda(-2-\lambda) - 2\cdot 4 = \lambda^2 + 2\lambda - 8$

$\rightsquigarrow \lambda = -1 \pm \sqrt{1+8} = -1 \pm 3 = -4, 2$

$\lambda = 2$ eigenvector: $\begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x_1 - x_2 = 0 \rightsquigarrow \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\rightsquigarrow$ homogeneous solution $e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -4$ eigenvector: $\begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ $2x_1 + x_2 = 0 \rightsquigarrow \underline{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\rightsquigarrow$ homogeneous solution $e^{-4t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

<u>Ansatz for particular solution</u>: $\underline{x}(t) = \begin{bmatrix} a \\ b \end{bmatrix}$

plug in: $\underline{x}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 & 2 \\ 4 & -2 \end{bmatrix}\underline{x} - \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2b - 2 \\ 4a - 2b - 6 \end{bmatrix}$

Solve: $\begin{matrix} 2b = 2 \\ 4a - 2b = 6 \end{matrix} \Leftrightarrow \begin{matrix} b = 1 \\ 4a = 6 + 2b = 8 \end{matrix} \Leftrightarrow \begin{matrix} b = 1 \\ a = 2 \end{matrix}$

plug in: $\underline{x}_p(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

<u>general solution</u>: $\underline{x}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-4t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

<u>initial value</u>: $\underline{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{matrix} 2 + c_1 - c_2 = 1 \\ 1 + c_1 + 2c_2 = 0 \end{matrix} \Leftrightarrow \begin{matrix} c_1 - c_2 = -1 \\ c_1 + 2c_2 = -1 \end{matrix} \Leftrightarrow \begin{matrix} c_1 = -1 \\ c_2 = 0 \end{matrix}$

$$\Rightarrow \underline{x}(t) = \underline{\underline{\begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}$$