MATH 54 FINAL EXAM (PRACTICE 1)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ______________________________

GSI's name: ______________________________

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Let $A = \begin{pmatrix} 1 & -1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$. Find a general solution to the homogeneous linear system with coefficient matrix A.

Solution:

$$\begin{pmatrix} 1 & -1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix} \leftarrow \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

$\Rightarrow$ General Solution is $\left\{ \begin{pmatrix} x_2 + x_6 \\ x_2 \\ -4x_6 \\ x_4 \\ -3x_6 \\ x_6 \end{pmatrix} \right\} = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -4 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right)$

(b) What is $Nullity(T_A)$? What is $Rank(T_A)$?

Solution:

$Rank(T_A) + Nullity(T_A) = 6$

$Nullity(T_A) = 3 \Rightarrow Rank(T_A) = 3$

2. (25 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T\begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\2\\0\end{pmatrix},\ T\begin{pmatrix}0\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix},\ T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-2\\0\\1\end{pmatrix}$$

Find the standard matrix of T. Is T one-to-one?

Solution:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 3 & -2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right) \leftarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & -1 & 0 & -3 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix}1\\0\\0\end{pmatrix} = 2\begin{pmatrix}1\\2\\-1\end{pmatrix} + 3\begin{pmatrix}0\\-1\\1\end{pmatrix} + (-1)\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} = (-1)\begin{pmatrix}1\\2\\-1\end{pmatrix} + (-2)\begin{pmatrix}0\\-1\\1\end{pmatrix} + 1\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = (-1)\begin{pmatrix}1\\2\\-1\end{pmatrix} + (-1)\begin{pmatrix}0\\-1\\1\end{pmatrix} + 1\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} T(\underline{e}_1) &= 2\begin{pmatrix}1\\2\\0\end{pmatrix} + 3\begin{pmatrix}1\\1\\2\end{pmatrix} + (-1)\begin{pmatrix}-2\\0\\1\end{pmatrix} \\ T(\underline{e}_2) &= (-1)\begin{pmatrix}1\\2\\0\end{pmatrix} + (-2)\begin{pmatrix}1\\1\\2\end{pmatrix} + 1\begin{pmatrix}-2\\0\\1\end{pmatrix} \\ T(\underline{e}_3) &= (-1)\begin{pmatrix}1\\2\\0\end{pmatrix} + (-1)\begin{pmatrix}1\\1\\2\end{pmatrix} + 1\begin{pmatrix}-2\\0\\1\end{pmatrix} \end{aligned}\right\} \Rightarrow A = \begin{pmatrix} 7 & -5 & -4 \\ 7 & -4 & -3 \\ 5 & -3 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -5 & -4 \\ 7 & -4 & -3 \\ 5 & -3 & -1 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix} \Rightarrow T \text{ one-to-one}$$

3. (25 points) Let A be a 4×3 matrix and B be a 3×2 matrix. Show that if the columns of A and B are linearly independent then the columns of AB are linearly independent. Hint: Consider the linear transformations associated to these matrices.

Solution:

Recall that given a matrix C, columns of C are L.I.

$\Leftrightarrow$ T_C one-to-one.

Columns of A, B L.I $\Rightarrow$ $T_A : \mathbb{R}^3 \to \mathbb{R}^4$ and $T_B : \mathbb{R}^2 \to \mathbb{R}^3$ are one-to-one

composition

$\Rightarrow T_A \circ T_B : \mathbb{R}^2 \to \mathbb{R}^4$ one-to-one

$\Rightarrow \underset{\displaystyle T_{AB}}{\overset{\shortparallel}{}}$ one-to-one $\Rightarrow$ Columns of AB are L.I.

4. (25 points) Let V be a vector space with bases $B = \{\underline{\mathbf{b}}_1, \underline{\mathbf{b}}_2, \underline{\mathbf{b}}_3\}$ and $C = \{\underline{\mathbf{c}}_1, \underline{\mathbf{c}}_2, \underline{\mathbf{c}}_3\}$, where

$$\underline{\mathbf{c}}_1 = \underline{\mathbf{b}}_1 - \underline{\mathbf{b}}_2, \qquad \underline{\mathbf{c}}_2 = \underline{\mathbf{b}}_1 + \underline{\mathbf{b}}_3, \qquad \underline{\mathbf{c}}_3 = \underline{\mathbf{b}}_1 + \underline{\mathbf{b}}_2 + \underline{\mathbf{b}}_3$$

If $(\underline{\mathbf{x}})_B = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, what is $(\underline{\mathbf{x}})_C$?

Solution:

$$P_{B \leftarrow C} = \left((\underline{c}_1)_B \ (\underline{c}_2)_B \ (\underline{c}_3)_B \right) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right) \leftarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right) \leftarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right)$$

$$P_{C \leftarrow B} = (P_{B \leftarrow C})^{-1} \Rightarrow P_{C \leftarrow B} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\Rightarrow (\underline{x})_C = P_{C \leftarrow B} (\underline{x})_B = \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$$

5. (25 points) Give an example of a non-diagonalizable matrix with only real eigenvalues. Carefully justify your answer.

Solution:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

algebraic multiplicity of 1

$$\det(A - x I_2) = \det \begin{pmatrix} 1-x & 1 \\ 0 & 1-x \end{pmatrix} = (1-x)^2 = 0 \Rightarrow x = 1$$

$$\mathrm{Nul}(A - 1 I_2) = \mathrm{Nul} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$\Rightarrow$ dim (1-eigenspace) $= 1 < 2 =$ algebraic multiplicity of 1

$\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ <u>not</u> diagonalizable

6. (25 points) Compute the minimum distance between $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ and

$$Nul \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 6 & 9 & 12 \\ 2 & 4 & 6 & 8 \end{pmatrix}$$

Solution:

$$W = Nul \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 6 & 9 & 12 \\ 2 & 4 & 6 & 8 \end{pmatrix} \Rightarrow W^{\perp} = Col \begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 6 \\ 4 & 12 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 6 \\ 4 & 12 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} \boxed{1} & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow W^{\perp} = Span \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$Proj_{W^{\perp}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \frac{-2}{30} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\Rightarrow \text{Min distance} = \left\| Proj_{W^{\perp}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\| = \frac{1}{15}\sqrt{30}$$

7. (25 points) Perform a singular-value decomposition of the matrix

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

Solution:

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} \Rightarrow A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}$$

$$\det(A^T A - x I_2) = (3-x)(12-x) - 36 = x^2 - 15x$$

$$\Rightarrow \lambda_1 = 15, \ \lambda_2 = 0 \qquad \Rightarrow \sigma_1 = \sqrt{15}, \ \sigma_2 = 0$$

$$\mathrm{Nul}(A^T A - 15 I_2) = \mathrm{Nul}\begin{pmatrix} -12 & 6 \\ 6 & -3 \end{pmatrix} = \mathrm{Span}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\mathrm{Nul}(A^T A - 0 I_2) = \mathrm{Nul}\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} = \mathrm{Span}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\underline{v}_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

$$\Rightarrow \underline{u}_1 = \frac{1}{\sigma_1} A \underline{v}_1 = \frac{1}{\sqrt{15}} \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} \\ \frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$\mathrm{Nul}(A^T) = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \mathrm{Span}\left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right)$$

Solution (continued) :

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix}$$

$$\left\| \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\| = \sqrt{2} \quad , \quad \left\| \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix} \right\| = \sqrt{3/2}$$

set $\underline{u}_2 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$, $\underline{u}_3 = \begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{15} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

8. (25 points) Find a general solution to the following differential equation

$$y'' - 2y' + y = 12t^2e^t$$

Solution:

$r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow$ 1 repeated auxiliary root

$\Rightarrow$ General solution to $y'' - 2y' + y = c_1 e^t + c_2 t e^t$

$$y_p(t) = t^2(A_0 + A_1 t + A_2 t^2)e^t = (A_0 t^2 + A_1 t^3 + A_2 t^4)e^t$$

$$\Rightarrow y_p'(t) = \left((2A_0 t + 3A_1 t^2 + 4A_2 t^3) + (A_0 t^2 + A_1 t^3 + A_2 t^4)\right)e^t$$

$$= \left(2A_0 t + (3A_1 + A_0)t^2 + (4A_2 + A_1)t^3 + A_2 t^4\right)e^t$$

$$\Rightarrow y_p''(t) = \left(2A_0 + (6A_1 + 2A_0)t + (12A_2 + 3A_1)t^2 + 4A_2 t^3\right)e^t$$

$$+ \left(2A_0 t + (3A_1 + A_0)t^2 + (4A_2 + A_1)t^3 + A_2 t^4\right)e^t$$

$$\Rightarrow y_p''(t) - 2y_p'(t) + y_p(t) =$$

$$\left(2A_0 + (6A_1 + 2A_0)t + (12A_2 + 3A_1)t^2 + \cancel{4A_2 t^3}\right)e^t$$

$$+ \left(2A_0 t + (3A_1 + A_0)t^2 + \cancel{(4A_2 + A_1)t^3} + \cancel{A_2 t^4}\right)e^t$$

$$- 2\left(2A_0 t + (3A_1 + A_0)t^2 + \cancel{(4A_2 + A_1)t^3} + \cancel{A_2 t^4}\right)e^t$$

$$+ \left(A_0 t^2 + \cancel{A_1 t^3} + \cancel{A_2 t^4}\right)e^t$$

Solution (continued) :

$$\Rightarrow \quad y''_p(t) - 2y'_p(t) + y_p(t) = \big((2A_0) + (6A_1)t + (12A_2)t^2\big)e^t$$

$$\Rightarrow \quad \begin{aligned} 2A_0 &= 0 \\ 6A_1 &= 0 \\ 12A_2 &= 12 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A_0 &= 0 \\ A_1 &= 0 \\ A_2 &= 1 \end{aligned}$$

$\Rightarrow$ General solution is $c_1e^t + c_2te^t + t^4e^t$

9. (25 points) Consider the following $\mathbb{R}^3$-valued function on $\mathbb{R}$:

$$\begin{pmatrix} t \\ t \\ t^2 \end{pmatrix}, \begin{pmatrix} t^2 \\ t^2 \\ t^3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix}.$$

Determine if these vector-valued functions are linearly independent. Are they solutions to some 3×3 homogeneous linear system of differential equations? Carefully justify your answers.

Solution:

$$c_1 \begin{pmatrix} t \\ t \\ t^2 \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ t^2 \\ t^3 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{for all } t \iff$$

$$c_1 t + c_2 t^2 + c_3 = 0 \quad \text{for all } t \iff c_1 = c_2 = c_3 = 0$$

$$\Rightarrow \begin{pmatrix} t \\ t \\ t^2 \end{pmatrix}, \begin{pmatrix} t^2 \\ t^2 \\ t^3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix} \text{ are L.I.}$$

However
$$\underbrace{\det \begin{pmatrix} t & t^2 & 1 \\ t & t^2 & 1 \\ t^2 & t^3 & t \end{pmatrix}}_{= W[\underline{x}_1, \underline{x}_2, \underline{x}_3](t)} = t(t^2 \cdot t - t^3) - t^2(t \cdot t - t^2) + 1(t \cdot t^3 - t^2 t^2)$$

$$= 0 \quad \text{for all } t$$

If $\underline{x}_1, \underline{x}_2, \underline{x}_3$ where solutions to a 3×3 homogeneous system then because L.I. $W[\underline{x}_1, \underline{x}_2, \underline{x}_3](t) \neq 0$ for all t.

This is not the case, hence there is no such 3×3 linear system.

10. (25 points) Calculate the sine Fourier series of the function $f(x) = x$, on the interval $[0, \pi]$. Use this to prove that

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots = \frac{\pi}{4}$$

Solution:

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin(nx)\, dx$$

$n \geq 1$

$$\int x \sin(nx)\, dx = \frac{-x}{n}\cos(nx) - \int 1 \cdot \frac{-1}{n}\cos(nx)\, dx$$

$$= \frac{-x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)$$

$$\Rightarrow \frac{2}{\pi}\int_0^{\pi} x \sin(nx)\, dx = \frac{2}{\pi}\left(\frac{-x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right)\Big|_0^{\pi}$$

$$= \frac{-2}{n}\left(\cos(n\pi)\right) - 0 = \frac{-2}{n}(-1)^n$$

$$\text{Fourier Sine Series} = \sum_{n=1}^{\infty} \frac{-2}{n}(-1)^n \sin(nx)$$

Evaluating $f(x) = x$ at $\frac{\pi}{2}$ gives

$$\Rightarrow \frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{-2}{n}(-1)^n \sin\left(\frac{n\pi}{2}\right)$$

$$= \frac{2}{1} - \frac{2}{3} + \frac{2}{5} - \frac{2}{7}$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$$

END OF EXAM