Name (Last, First): ______________________

Student ID ______________________

Circle your section:

301	MWF 8-9A	121 LATIMER	LIANG
303	MWF 9-10A	121 LATIMER	SHAPIRO
306	MWF 10-11A	237 CORY	SHAPIRO
307	MWF 11-12P	736 EVANS	WORMLEIGHTON
309	MWF 4-5P	100 WHEELER	RABINOVICH
313	MWF 2-3P	115 KROEBER	LIANG
314	MWF 1-2P	110 WHEELER	WORMLEIGHTON
315	MWF 3-4P	121 LATIMER	RABINOVICH

If none of the above, please explain: ______________________

Only this exam and a pen or pencil should be on your desk.

(You can get scratch paper from me if you need it.)

Problem	Points Possible	Your Score
A	10	
B	10	
C	10	
D	10	
E	10	
F	10	
G	10	
H	10	
I	10	
J	10	

Problem A. Decide if the following are **always true** or **at least sometimes false**. Enter your answers as **T** or **F** in the following chart. Correct answers receive 1 points, incorrect answers receive -1 points, and blank answers receive 0 points. No justification is necessary, although if you believe the question is ambiguous, record your interpretation below it. A is always a matrix.

Statement	1	2	3	4	5	6	7	8	9	10
Answer										

1. The equation $y''(t) = (y'(t))^2$ is a differential equation. True. (Although it's not linear.)
2. Every linear, constant coefficient, homogenous ODE has a basis of solutions of the form e^{ct} for varying c. False. (Consider e.g. $y''(t) = 0$, or more generally any equation for which the auxilliary polynomial has a multiple root.)
3. If the Wronskian of a collection of functions vanishes at any point, then the functions are linearly dependent. False. (This is only true if the functions are solutions to a linear ODE whose order is the same as the number of functions being considered.)
4. There exists a unique solution to the equation $y''(t) + \cos(t)y'(t) + ty(t)$ with $y(0) = 1$ and $y'(0) = 2$. True. (Existence and uniqueness theorem.)
5. There is a third-order, linear, constant coefficient, homogenous ODE with t^4 as a solution. False. (A linear constant coefficient homogenous ODE with t^4 as a solution must have 0 as a multiplicity 5 root of its auxilliary polynomial, so must have degree at least 5.)
6. The motion of a spring is described by a linear ordinary differential equation. True. (It's $x''(t) + kx(t) = 0$.)
7. The heat equation is a linear differential equation. True.
8. Computing Fourier coefficients can be thought of as an orthogonal projection. True.
9. The Fourier expansion of the function $|x|$ on the interval $[-\pi, \pi]$ has no cosine terms. False. (The function $|x|$ is even, so one should expect cosine terms.)[1]
10. The space of f satisfying $\int_0^{10} f(x)dx = 0$ is a vector space. True.

[1] Incidentally this question is not completely precise: before asking for "the Fourier expansion" one should specify how exactly a function is made periodic. I will do so on the exam. However in this case, no matter how you make this function periodic, you will need cosine terms in the Fourier expansion, just because of the behavior near 0.

Problem B. Give an example, or explain why none exists.

1. (3 pts) A linear partial differential equation.

One could make a case that $0 = 0$ is a linear PDE in which all the coefficients happen to be zero, I would have given credit to someone who wrote this and argued for it. However a more reasonable answer is e.g. the heat equation

$$\frac{\partial}{\partial t} f(x,t) = \frac{\partial^2}{\partial x^2} f(x,t)$$

2. (3 pts) A linear, constant coefficient, homogenous, second order ODE with solutions e^{2x}, e^x.

We should have the auxilliary polynomial $(z-1)(z-2)$, hence the ODE

$$\left(\frac{d}{dx} - 1\right)\left(\frac{d}{dx} - 2\right) f(x) = 0$$

Expanding it out gives $f''(x) - 3f'(x) + 2f(x) = 0$.

3. (4 pts) A third order linear, constant coefficient, homogenous ODE with solutions x, e^x.

We need 0 to be a double root of the auxilliary polynomial, and 1 a simple root. So the auxilliary polynomial is $z^2(z-1)$, and the ODE is

$$\left(\frac{d}{dx}\right)^2 \left(\frac{d}{dx} - 1\right) f(x) = 0$$

or expanding it out, $f'''(x) - f''(x) = 0$.

Problem C.

(1 pt) Give the general solution to the equation $y' = 3y$.

$y = ce^{3t}$, some constant c.

(2 pts) Give the general solution to the equation $y'' + 3y' + 2y = 0$.

$y = ae^{-t} + be^{-2t}$, some constants a, b.

(3 pts) Give the general solution to the equation $y'' + 4y = 0$. Be sure to use real valued functions.

$y = a\sin(2t) + b\cos(2t)$, some constants a, b.

(4 pts) Give the general solution to the equation $y'' + 3y' + 2y = e^{-2t}$.

The auxilliary equation is $(z+1)(z+2)$, of which -2 is a simple root. So we should try te^{-2t}. We find

$$\left(\frac{d}{dt} + 1\right)\left(\frac{d}{dt} + 2\right) te^{-2t} = \left(\frac{d}{dt} + 1\right) e^{-2t} = -2e^{-2t} + e^{-2t} = -e^{-2t}$$

So, $-te^{-2t}$ gives one solution. We already found the general solution for the homegenous part above, and we add these to get

$$y = -te^{-2t} + ae^{-t} + be^{-2t}$$

Problem D.

(10 pts) State the existence and uniqueness theorem for linear homogenous ODE. (You can use any of the various equivalent formulations.)

An order n linear homogenous ODE for the unknown function $y(t)$ has a unique solution with any prescribed $y(t_0), y'(t_0), \ldots, y^{(n-1)}(t_0)$, for any t_0 in the domain of definition of the coefficients, which are assumed to be continuous functions of t.

Problem E. Consider the equation $y''(t) - 4y(t) = 0$.

(3 pts) Find a basis for the solutions

e^{2t}, e^{-2t}

(3 pts) Compute the Wronskian of the basis you found

$$\det \begin{bmatrix} e^{2t} & e^{-2t} \\ 2e^{2t} & -2e^{-2t} \end{bmatrix} = -4$$

(4pts) Find $y(t)$ satisfying the above equation, such that $y(0) = 1$ and $y'(0) = 1$.

We must solve

$$\begin{bmatrix} e^{2t} & e^{-2t} \\ 2e^{2t} & -2e^{-2t} \end{bmatrix}_{t=0} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Inverting the matrix,

$$\begin{bmatrix} a \\ b \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$$

Thus the solution is

$$y(t) = \frac{3}{4}e^{2t} + \frac{1}{4}e^{-2t}$$

Problem F. Consider the equation $y'''(t) - 2y''(t) + y'(t) - 2y(t)$.

(5pts) Find a basis of real solutions

The auxilliary polynomial is $z^3 - 2z^2 + z - 2 = (z-2)(z+i)(z-i)$, so a basis of real solutions is $e^{2t}, \sin(t), \cos(t)$.

(5pts) Solve the initial value problem $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$.

In terms of the fundamental solution matrix we want to solve:

$$\begin{bmatrix} e^{2t} & \sin(t) & \cos(t) \\ 2e^{2t} & \cos(t) & -\sin(t) \\ 4e^{2t} & -\sin(t) & -\cos(t) \end{bmatrix}_{t=0} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Probably this linear equation is easier to solve by just looking than by inverting the matrix; in any case the solution is $a = -c = 2/5$ and $b = 1/5$. We then have

$$y(t) = \frac{2}{5}e^{2t} + \frac{1}{5}\sin(t) - \frac{2}{5}\cos(t)$$

Problem G.

(2pts) Compute $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^4, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^5$

These are, respectively, $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 5 & 8 \end{pmatrix}$

(5pts) Compute the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

The characteristic polynomial is $-x(1-x) - 1 = x^2 - x - 1$. The roots of this are $x = \frac{1 \pm \sqrt{5}}{2}$. The corresponding eigenvectors are the null spaces for $\begin{pmatrix} \frac{-1 \mp \sqrt{5}}{2} & 1 \\ 1 & \frac{1 \mp \sqrt{5}}{2} \end{pmatrix}$, namely the vectors $(2, 1 \pm \sqrt{5})^T$.

(3pts) Compute $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{100}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{100} = -\frac{1}{4\sqrt{5}} \begin{pmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}^{100} \begin{pmatrix} 1-\sqrt{5} & -2 \\ -1-\sqrt{5} & 2 \end{pmatrix}$$

Problem H.

(10 pts) Find bases for the kernel and image of the linear transformation given by the matrix:

$$M = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

The organized way to do this involves row reduction. Let me do something different. Just by looking, one sees the following relations among rows: $2r_2 = r_1 + r_3$ and $2r_3 = r_2 + r_4$. That is, the vectors $(1, -2, 1, 0)^T, (0, 1, -2, 1)^T$ are in the kernel (aka null space). They're linearly independent, so the kernel is at least 2 dimensional. It's also at most two dimensional, since if it were 3 dimensional, then the row span would be 1 dimensional, so all rows would be multiples of each other. That's not the case. So, those two elements give a basis.

Now for the image. This is the same as the column span. The columns satisfy the same relations, $2c_2 = c_1 + c_3$ and $2c_3 = c_2 + c_4$. From this one sees easily that c_2, c_3 can be written in terms of c_1, c_4. As c_1, c_4 are not multiples, they're linearly independent, hence the first and fourth columns form a basis of the column space.

Problem I.

(10pts) Consider the function $f(x)$ defined on $[0, \pi]$ by the formula

$$f(x) = \begin{cases} -x & x < \pi/2 \\ \pi - x & x \geq \pi/2 \end{cases}$$

Determine the sine Fourier series of this function.

We should compute the coefficients by

$$b_n = \frac{2}{\pi}\left(\int_0^{\pi/2} -x\sin(nx)dx + \int_{\pi/2}^{\pi} (\pi - x)\sin(nx)dx\right) =$$

$$\frac{2}{\pi}\left(\int_0^{\pi} -x\sin(nx)dx + \int_{\pi/2}^{\pi} \pi\sin(nx)dx\right) =$$

$$\frac{2}{\pi}\left(\frac{1}{n^2}\int_0^{n\pi} -u\sin(u)du + \frac{\pi}{n}\int_{n\pi/2}^{n\pi} \sin(u)du\right) =$$

$$\frac{2}{\pi}\left(\frac{1}{n^2}[u\cos(u) - \sin(u)|_0^{n\pi} + \frac{\pi}{n}[-\cos(u)|_{n\pi/2}^{n\pi}\right) =$$

$$= \frac{2}{n}\cos(n\pi/2)$$

Noting that the above vanishes unless $n = 2k$ is even,

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k}(-1)^k \sin(2kx)$$

Problem J.

(10 pts) Consider a wire of length π, which is stretched from $x = 0$ to $x = \pi$.

Suppose the initial temperature is given by the function

$$u(x,0) = \begin{cases} 0 & x < \pi/2 \\ \pi & x \geq \pi/2 \end{cases}$$

and that as time progresses, the ends are kept at the temperatures 0 and π respectively.

Using these initial and boundary conditions, solve the heat equation

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)$$

Subtracting off the stationary solution $u(x,t) = x$, we are reduced to solving the heat equation with $u(x,0) - x$ given by the function of the previous question. We have already found its sine Fourier series. It remains only to restore the appropriate eigenfunctions of $\partial/\partial t$ to write the solution:

$$u(x,t) = x + \sum_{k=1}^{\infty} \frac{1}{k}(-1)^k \sin(2kx) e^{-(2k)^2 t}$$