Problem A. Decide if the following are **always true** or **at least sometimes false**. Enter your answers by filling the bubbles provided. Correct answers receive 1 points, incorrect answers receive -1 points, and blank answers receive 0 points. No justification is necessary, although if you believe the question is ambiguous, record your interpretation below it.

1. If A, B are matrices of the same size and rank, then there exist square matrices C, D such that $CAD = B$.

   True

2. If A is a square matrix, then there is some invertible matrix C such that CAC^{-1} is diagonal.

   False

3. It is possible that one basis of a vector space has 2 elements, and a different basis of the same space has 3 elements.

   False

4. If V is a subspace of $\mathbb{R}^n$, then the dimension of V is at most n.

   True

5. It is possible to have mutually orthogonal nonzero vectors v_1, v_2, v_3 in $\mathbb{R}^2$. I.e., vectors such that $v_i \cdot v_j = 0$ whenever $i \neq j$.

   False

6. Every subspace of $\mathbb{R}^n$ has an orthonormal basis.

   True

7. The change of basis matrix to go from coordinates in a basis $\mathcal{B}$ to those in a basis $\mathcal{C}$ is the inverse of the change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$.

   True

8. It is sometimes possible to find a nonzero vector inside the intersection of a subspace and its orthogonal complement.

   False

9. Eigenvectors with distinct eigenvalues are linearly independent.

   True

10. Any two vector spaces of the same (finite) dimension are isomorphic.

    True

Problem B. Given an example — **and explain why it is an example** — or explain why none can exist.

Answers should be given in complete sentences. A sentence begins with a capital letter, has a subject and verb, and ends with a period. You **will** *lose points otherwise.*

1. (3 pts) A 2x2 real matrix with no real eigenvalues

Consider the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The eigenvalues must be roots of the characteristic equation $x^2 + 1 = 0$, which has no real roots. In fact the eigenvalues are $\pm i$. (Other examples include: the matrix of any rotation; this one is rotation by $90°$.)

Aside: if you think of $\mathbb{C}$ as a 2 dimensional real vector space, and choose the basis $1, i$, then this is the matrix of the linear transformation "multiplication by i".

2. (3 pts) A 3x3 real matrix with no real eigenvalues

No such matrix exists. Any root of the characteristic polynomial gives an eigenvalue, and the characteristic polynomial of an 3x3 (or more generally, odd dimensional) square has odd degree, and therefore must have a real root.

3. (4 pts) Two matrices with the same characteristic polynomial, exactly one of which has a basis of eigenvectors (say which one!)

Consider the matrices $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Both matrices have characteristic polynomial x^2, and therefore only have zero as an eigenvalue. The second is the zero matrix, so any vector is an eigenvector of eigenvalue zero, and in particular, there is a basis of eigenvectors. For the first matrix, note that the zero eigenspace is the same as the null space, which is spanned by the second standard basis vector; this being the only eigenspace, there cannot be a basis of eigenvectors.

Problem C.

Consider the matrix $A = \begin{bmatrix} 5 & -3 \\ 1 & 1 \end{bmatrix}$

1. (2 pts) Find the eigenvalues of A.

The characteristic equation is $0 = (5-x)(1-x)-(-3) = x^2 - 6x + 8 = (x-4)(x-2)$. So the eigenvalues are 2 and 4.

2. (3 pts) Find the eigenvectors of A

These will span the null spaces of $A - 2I = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}$, $A - 4I = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}$.

By inspection, an eigenvector of eigenvalue 2 is $(1,1)^T$ and for eigenvalue 4 is $(3,1)^T$.

3. (3 pts) Write an equation $A = XDX^{-1}$ some explicit matrices X, X^{-1} and diagonal D.

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -1/2 & 3/2 \\ 1/2 & -1/2 \end{bmatrix}$$

4. (2 pts) Compute A^{1000}.

$$A^{1000} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{1000} & 0 \\ 0 & 4^{1000} \end{bmatrix} \begin{bmatrix} -1/2 & 3/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2^{999} & 3 \cdot 2^{999} \\ 2 \cdot 4^{999} & -2 \cdot 4^{999} \end{bmatrix}$$

$$= \begin{bmatrix} -2^{999} + 6 \cdot 4^{999} & 3 \cdot 2^{999} - 6 \cdot 4^{999} \\ -2^{999} + 2 \cdot 4^{999} & 3 \cdot 2^{999} - 2 \cdot 4^{999} \end{bmatrix}.$$

(Sanity check: replacing $1000 - 1 = 999$ with $1 - 1 = 0$, we recover the original matrix)

Problem D. Consider the bases

$$\mathcal{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \mathcal{C} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

1. (5 pts) Determine the change of basis matrix which converts *from* coordinates in the basis $\mathcal{B}$ *to* coordinates in the basis $\mathcal{C}$. (In your book this matrix would be called $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.)

2. (5 pts) Explain in your own words why the procedure you used above works.

I'll solve this problem and explain why it works as I go along (answering both parts at once).

First observe that we can get from $\mathcal{B}$ to $\mathcal{C}$ by first going from $\mathcal{B}$ to the standard basis, and then going from the standard basis to $\mathcal{C}$. Second, note that going from the standard basis to $\mathcal{C}$ is the inverse of going from $\mathcal{C}$ to the standard basis.

That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{C}\leftarrow std}{P}\right)\left(\underset{std\leftarrow\mathcal{B}}{P}\right) = \left(\underset{std\leftarrow\mathcal{C}}{P}\right)^{-1}\left(\underset{std\leftarrow\mathcal{B}}{P}\right)$$

It remains to explain how to get the matrix to change to the standard basis. Let's recall in coordinates with respect to a basis $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$, the expression $\begin{bmatrix} x \\ y \end{bmatrix}_{\mathcal{D}}$ means $x\mathbf{d}_1 + y\mathbf{d}_2$. So to change coordinates to the standard basis, we need a matrix $\underset{std\leftarrow\mathcal{D}}{P}$ with the effect that $\underset{std\leftarrow\mathcal{D}}{P}\begin{bmatrix} x \\ y \end{bmatrix}$ is the expression of $x\mathbf{d}_1 + y\mathbf{d}_2$ in the standard basis. In particular, it must be that $\underset{std\leftarrow\mathcal{D}}{P}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{d}_1$ and $\underset{std\leftarrow\mathcal{D}}{P}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{d}_2$. The columns of a matrix are given by where the matrix sends the standard basis, hence $\underset{std\leftarrow\mathcal{D}}{P} = [\mathbf{d}_1 \quad \mathbf{d}_2]$.

Now we finish the computation: $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{C}\leftarrow std}{P}\right)\left(\underset{std\leftarrow\mathcal{B}}{P}\right) = \left(\underset{std\leftarrow\mathcal{C}}{P}\right)^{-1}\left(\underset{std\leftarrow\mathcal{B}}{P}\right) =$

$$= \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

Some of you will have made the 2x4 matrix $\left[\underset{std\leftarrow\mathcal{C}}{P} \middle| \underset{std\leftarrow\mathcal{B}}{P}\right]$, row reduced it, and taken the right part. In this case you should also explain why this is computing $\left(\underset{std\leftarrow\mathcal{C}}{P}\right)^{-1}\left(\underset{std\leftarrow\mathcal{B}}{P}\right)$. ("for the same reason that row reduction lets you compute the inverse of a matrix" would be a good enough answer.)

Problem E.

(10 pts) Produce an orthonormal basis for the orthogonal complement to the vector

$$v = (1, -1, 1, -1)$$

Show your work and explain briefly in words what you are doing in each step.

There are many ways to do this problem. Probably most of you did the following: first, find a basis for the orthogonal complement by thinking of v as a linear equation — you get the basis $(1, 1, 0, 0), (-1, 0, 1, 0), (1, 0, 0, 1)$ — then apply Gram-Schmidt. Another possibility: you can skip the first step, complete v to a basis *in any way you want*, then apply Gram-Schmidt; since the resulting basis will consist of a vector proportional to v, and then three vectors orthogonal to it. These are both good ways to solve the problem, and really the only way to go about it for a totally arbitrary vector v.

But I personally would have done something else. The vector v is pretty simple, and one can see right away very many vectors orthogonal to it, e.g., $(1, 1, 0, 0)$, $(0, 1, 1, 0)$, $(0, 0, 1, 1)$, $(1, 0, 0, 1)$, $(1, 0, -1, 0)$, $(0, 1, 0, -1)$, and many more. So I look at these vectors; it's clear (look where the zeroes are) they span at least a 3-dimensional space, hence the orthogonal complement. So I try and just pick out an orthogonal basis out of them. Well, $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$ are obviously orthogonal, so start with those two. $(1, 0, 0, 1)$ and $(0, 1, 1, 0)$ aren't orthogonal to these, but their difference is. So $(1, 1, 0, 0), (0, 0, 1, 1), (1, -1, -1, 1)$ is an orthogonal basis for the orthogonal complement, and normalizing them gives

$$\frac{1}{\sqrt{2}}(1, 1, 0, 0), \frac{1}{\sqrt{2}}(0, 0, 1, 1), \frac{1}{2}(1, -1, -1, 1)$$

This guess and check "method" isn't guaranteed to work, and certainly is a bad choice for an arbitrary problem. But if you do manage to guess enough mutually orthogonal vectors, then you necessarily have produced the desired basis.