EECS 16B Designing Information Devices and Systems II
Spring 2019 UC Berkeley

Final Exam

Exam location: GPB 100

PRINT your student ID: ____

PRINT AND SIGN your name: __________________________________, ______________________
(first and last) (signature)

PRINT your discussion section and GSI(s) (the one(s) you attend): ____

Row Number (front row is 1): __________ Seat Number (left most is 1): __________

Name and SID of the person to your left: ____

Name and SID of the person to your right: ____

Name and SID of the person in front of you: ____

Name and SID of the person behind you: ____

Section 0: Pre-exam questions (3 points)

1. **What has been your favorite course at UC Berkeley so far? (1 pt)**

2. **What would you do on your perfect vacation? Describe how you would feel. (2 pts)**

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

PRINT your name and student ID: ____

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

PRINT your name and student ID: ________________________________

3. PCA (14 pts)

In this problem, we are going to think of our data points as being given in **columns**. You can imagine that the data points are recordings from a microphone. We take many such recordings. Our goal is to identify the principal components so that we could, in the future, project fresh recordings from the microphone onto those principal components to help us better understand what was being said.

(a) **(2 pts)** Suppose for this part, that you have four observed data vectors (say corresponding to the same spoken word, being repeated four times) and all of them just happened to be multiples of the following 6-dimensional vector $\vec{v} = \begin{bmatrix} 3 \\ -4 \\ 5 \\ -5 \\ 4 \\ -3 \end{bmatrix}$. (For your convenience, note that $\|\vec{v}\| = 10$.)

You arrange the data vectors as the columns of a matrix A given by:

$$A = \begin{bmatrix} | & | & | & | \\ -\vec{v} & -2\vec{v} & \vec{v} & 2\vec{v} \\ | & | & | & | \end{bmatrix} \tag{1}$$

You want to perform PCA to better understand your data. **Find the first principal component vector of A to explain the nature of your data points.**

(HINT: You don't need to compute any covariance matrices or compute any eigenvalue/eigenvectors in this simple case. Also, be sure to think about what size vector you want as the answer. Don't forget to normalize!)

Solution:

Principal component analysis is in general about understanding how best to approximate our (potentially) high-dimensional data (like recordings from a microphone) with its lower-dimensional essence. The first principal component is about seeing which one-dimensional line best approximates the data points — i.e. which is the line for which projecting the data points onto it results in "estimates" that are as close as possible to the data points.

In the case of this problem, every point is explicitly given as a multiple of a single vector $\vec{v}$ and so the data already lies on a straight line going through the origin. So, the first principal component is just along the direction of $\vec{v}$. Because a principal component represents a direction, it is conventional to normalize the vector to have unit length. In this case, we are told that the vector $\vec{v}$ has length 10, and so the answer is $\frac{\vec{v}}{10}$.

(Because the line is all that matters, you could also have used the negative of this $-\frac{\vec{v}}{10}$.)

A more methodical way to do PCA is to invoke the SVD. First, however, you need to make sure that your data is zero-mean because the SVD will only give you directions relative to the origin. In this problem, all the data is zero-mean by construction.

The singular value decomposition of a matrix A is a way of decomposing A into a sum of rank 1 matrices. In this sum the i^{th} rank 1 matrix is formed from taking the outer product of normalized column vectors $\vec{u}_i$ and normalized row vectors $\vec{v}_i^T$, scaled by their respective singular values σ_i.

(Note that the $\vec{v}_i^T$ row vectors in the SVD decomposition $A = U\Sigma V^T$ are completely unrelated to the $\vec{v}$ column vector that we have defined for our data matrix A above.)

Looking at our given A, we can see that the matrix itself is rank 1 as the columns are all multiples of the same vector: $\vec{v}$. Seeing this we realize we can rewrite the matrix A as the following outer product:

$$A = \begin{bmatrix} | \\ \vec{v} \\ | \end{bmatrix} \begin{bmatrix} -1 & -2 & 1 & 2 \end{bmatrix} \tag{2}$$

However the SVD requires we normalize the vectors $\vec{u}_1$ and $\vec{v_1^T}$. In order to reconstruct A properly we must scale back with the norms that we divided out to normalize.

$\|[-1,-2,1,2]\| = \sqrt{10}$ and $\|\vec{v}\| = 10$. Consequently, when we pull that out, we get $\sigma_0 = 10\sqrt{10}$ as the singular value that corresponds to the first (and only) principal component.
Thus we can write the SVD of A as:

$$A = \begin{bmatrix} | \\ \frac{\vec{v}}{10} \\ | \end{bmatrix} 10\sqrt{10} \begin{bmatrix} \frac{-1}{\sqrt{10}} & \frac{-2}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \end{bmatrix} \tag{3}$$

Now we just have to pick which normalized vector to deem the principal component. Since our data (the microphone recordings) are collected as columns we choose $\frac{\vec{v}}{10}$ as the principal component.
How could you have remembered that you had to use columns here? The reason is that you were told that you wanted to project fresh recordings from the microphone onto the principal component. You have to project onto a vector of the same size. Only one of the vectors is the right size.

(b) **(6 pts)** Suppose that now, we have two more data points (say, corresponding to a *different* word being spoken twice) that are multiples of a different vector $\vec{p}$ where:

$\vec{p} = \begin{bmatrix} \frac{36}{\sqrt{134}} \\ -\frac{8}{\sqrt{134}} \\ -\frac{28}{\sqrt{134}} \\ 0 \\ 0 \\ 0 \end{bmatrix}$. (For your convenience, note that $\|\vec{p}\| = 4$ and that $\vec{p}^T\vec{v} = 0$.)

We augment our data matrix with these two new data points to get:

$$A = \begin{bmatrix} | & | & | & | & | & | \\ -\vec{v} & -2\vec{v} & \vec{v} & 2\vec{v} & -4\vec{p} & 4\vec{p} \\ | & | & | & | & | & | \end{bmatrix} \tag{4}$$

Find the principal components corresponding to the nonzero singular values of A. **What is the first principal component vector? What is the second principal component vector? Justify your answer.**

(Hint: Think about the inner product of $\vec{v}$ and $\vec{p}$ and what that implies for being able to appropriately decompose A. Again, very little computation is required here.)

Solution: The solution to the previous part tells you what we need to do. We need to find the best two-dimensional subspace that best represents our data.

We start by taking the SVD of A.

The columns of A are all multiples of two vectors: $\vec{v}$ and $\vec{p}$. Each of these can be used to create a rank 1 matrix, and these can be summed together to form A.

Since $\vec{v}$ and $\vec{p}$ are orthogonal to one another, our life is easier. This problem's A matrix is made especially nice by seeing that a data point is either purely in the $\vec{v}$ direction, or the $\vec{p}$ direction.

Using this knowledge we rewrite A as:

$$A = \begin{bmatrix} | \\ \vec{v} \\ | \end{bmatrix} \begin{bmatrix} -1 & -2 & 1 & 2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} | \\ \vec{p} \\ | \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & -4 & 4 \end{bmatrix}.$$

The orthogonality relationships demanded by the SVD are clearly satisfied since the row-vectors involved above have disjoint support (i.e. when one is nonzero, the other is zero) and the columns are orthogonal since we've been told so.

However for the SVD the vectors: $\vec{u}_1$, $\vec{v_1}^T$, $\vec{u}_2$ and $\vec{v_2}^T$ must be normalized and each rank 1 matrix must be scaled by the appropriate σ_i to allow the sum to properly reconstruct A. We also need to figure out which σ_i is bigger so we can order them properly. In the previous part, we have already done the calculations for $\vec{v}$'s part in this story. So what remains is the $\vec{p}$ part. Clearly the norm of the relevant row is $4\sqrt{2}$ which the norm of the relevant column is 4. So the singular value in question is $16\sqrt{2}$.

Using this we can rewrite A as:

$$= \begin{bmatrix} | \\ \frac{\vec{v}}{10} \\ | \end{bmatrix} 10\sqrt{10} \begin{bmatrix} \frac{-1}{\sqrt{10}} & \frac{-2}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} & 0 & 0 \end{bmatrix} + \begin{bmatrix} | \\ \frac{\vec{p}}{4} \\ | \end{bmatrix} 16\sqrt{2} \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{-4}{4\sqrt{2}} & \frac{4}{4\sqrt{2}} \end{bmatrix}.$$

So, what is bigger $10\sqrt{10}$ or $16\sqrt{2}$? The easiest thing to do is just to square each of them and compare $10^3 = 1000$ to $2^9 = 512$.

From this we see that our singular values are $\sigma_1 = 10\sqrt{10}$ and $\sigma_2 = 16\sqrt{2}$ since $10\sqrt{10} > 16\sqrt{2}$. Thus $\frac{\vec{v}}{10}$ which corresponds to σ_1 is still the first principal component vector and $\frac{\vec{p}}{4}$ which corresponds to σ_2 is the second principal component vector.

(c) **(6 pts)** In the previous part, you had

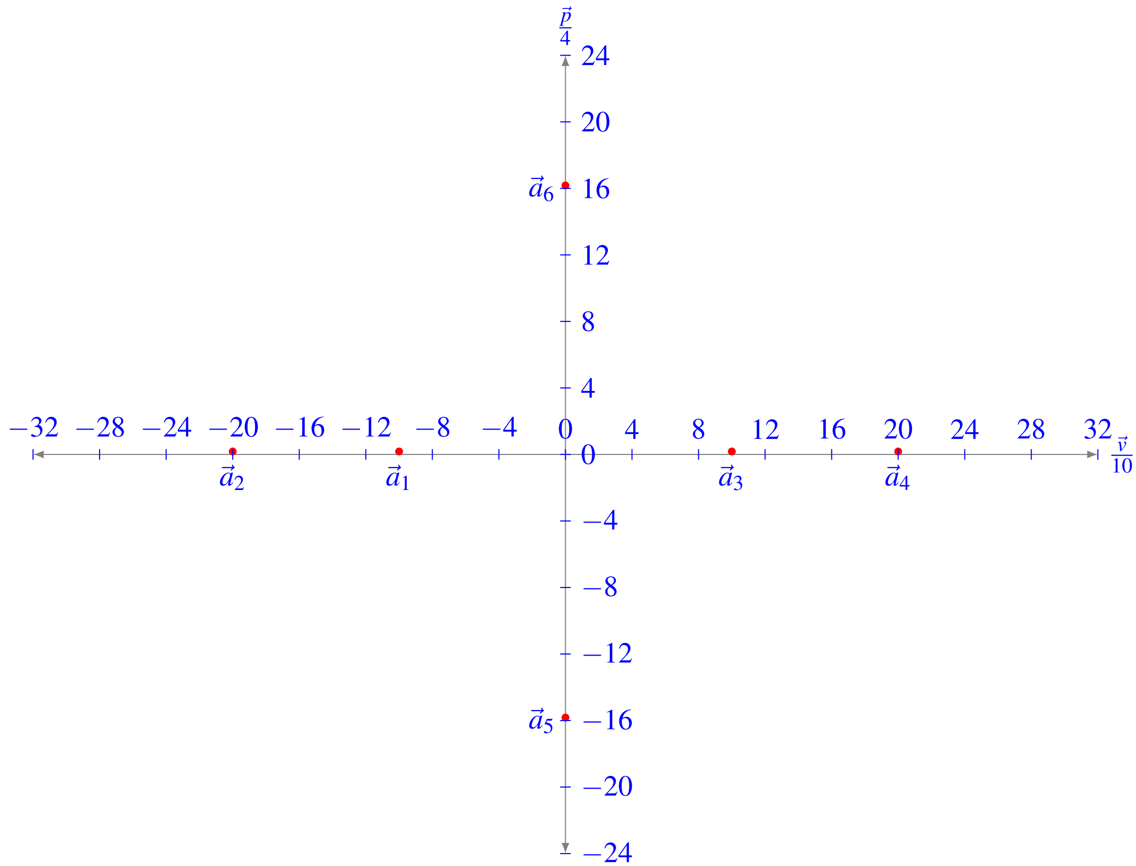
$$A = \begin{bmatrix} | & | & | & | & | & | \\ -\vec{v} & -2\vec{v} & \vec{v} & 2\vec{v} & -4\vec{p} & 4\vec{p} \\ | & | & | & | & | & | \end{bmatrix}$$

with $\|\vec{v}\| = 10$ and $\|\vec{p}\| = 4$, satisfying $\vec{p}^T\vec{v} = 0$.

If we use $\vec{a}_i$ to denote the i-th column of A, **plot the data points $\vec{a}_i$ projected onto the first and second principal component vectors.** The coordinate along the first principal component should be represented by horizontal axis and the coordinate along the second principal component should be the vertical axis. **Label each point**.

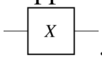
Solution:

Once we know what the principal components are, we know that the first four data points are just multiples of the first principal component and the last two data points are just multiples of the second principal component. What multiples? For the first four, the multiples are clearly $-10, -20, 10, 20$ since the norm of $\vec{v}$ is 10. For the final two, the multiples are clearly $-16, +16$ since the norm of $\vec{p}$ is 4. Plotting:

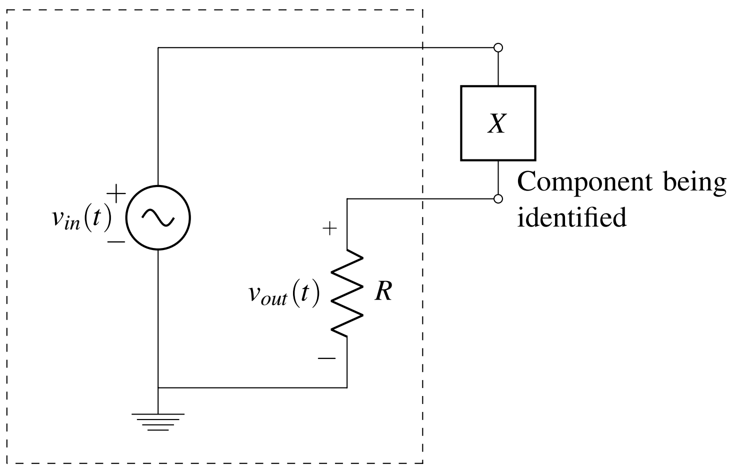


PRINT your name and student ID: ____

4. Identifying an Unknown Circuit Component (24 pts)

Suppose we have an unknown circuit component, which we'll denote as X and represent with the symbol . X could either be a resistor, a capacitor, or an inductor, but we don't know which one it is, nor do we know what its *component value* (that is, its resistance, capacitance, or inductance) could be. If you needed to identify X, that is figure out what kind of component X is and figure out its value, you would use a tool called an *RLC meter*. **In this problem, you will examine how an RLC meter can identify unknown circuit components with the help of transfer functions and the DFT**.

In circuit form, an RLC meter looks like this:



RLC meter

Here, $v_{in}(t) = A_{in}\cos(2\pi f_0 t + \theta_{in})$ is a known sinusoidal test input of known frequency f_0, known amplitude A_{in}, and known phase θ_{in}; while R is also a known resistance. Under this setup, we know that $v_{out}(t)$ will also be a sinusoid, which we'll denote as $v_{out}(t) = A_{out}\cos(2\pi f_0 t + \theta_{out})$.

When X is connected to the RLC meter, an on-board microcontroller takes samples from $v_{in}(t)$ and $v_{out}(t)$ and uses these samples to compute $Z_X|_{f_0}$, the impedance of the unknown component at frequency f_0. From the value of $Z_X|_{f_0}$, it can figure out whether X is a resistor, a capacitor, or an inductor, as well as what resistance, capacitance, or inductance it has.

(a) **(4 pts) Find the transfer function $H(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}}$ when the unknown component is connected to the RLC meter**. Here, $\widetilde{V}_{out}$ and $\widetilde{V}_{in}$ denote the phasor representations of $v_{out}(t)$ and $v_{in}(t)$, respectively. Answer in terms of R, the known resistance, and $Z_X(\omega)$, the unknown impedance.

Solution: With component X in place, the portion of the RLC meter we've shown you is a *voltage divider*, with impedances R and Z_X. With that in mind, the transfer function is

$$H(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} = \left(\frac{R}{R+Z_X}\widetilde{V}_{in}\right) \times \frac{1}{\widetilde{V}_{in}} = \frac{R}{R+Z_X}. \tag{5}$$

(b) **(4 pts)** Suppose that we know $H(\omega_0)$, that is the (possibly complex) numerical value of $H(\omega)$ at the angular frequency $\omega_0 = 2\pi f_0$. **Show how to use the value of $H(\omega_0)$ to calculate $Z_X|_{f_0}$, the impedance of the unknown component at the frequency** f_0. Your result should be an equation for $Z_X|_{f_0}$ in terms of quantities whose values we know.

Solution: From part (a) we have an expression for $H(\omega)$ in terms of Z_X. If we take this expression at the specific angular frequency ω_0, we get

$$H(\omega_0) = \frac{R}{R + Z_X|_{f_0}}. \tag{6}$$

If we solve this expression for $Z_X|_{f_0}$, we get

$$Z_X|_{f_0} = \frac{R}{H(\omega_0)} - R = R\left(\frac{1}{H(\omega_0)} - 1\right). \tag{7}$$

Since we know the value of R, and we are given $H(\omega_0)$ for this part, this equation is what we wanted to find: an equation for $Z_X|_{f_0}$ in terms of known quantities.

(c) **(6 pts)** Suppose that we knew $Z_X|_{f_0}$. **Describe how to use $Z_X|_{f_0}$ to determine both what kind of component X is and the corresponding component value?** *(HINT: Physical resistances, capacitances, and inductances are always positive. And $\frac{1}{j} = -j$ for $j = \sqrt{-1}$.)*

Solution: To figure out if X is a resistor, capacitor, or inductor, it would suffice to look at the *phase* of $Z_X|_{f_0}$. If the phase were positive, then X would have to be an inductor; if it were negative, X would have to be a capacitor; and if it were zero, then X would have to be a resistor. Once we've decided what kind of component X is this way, we can use the magnitude of $Z_X|_{f_0}$ to determine the value:

$$C_X = \frac{1}{2\pi f_0 |Z_X|_{f_0}|}, \text{ if } X \text{ is a capacitor} \tag{8}$$

$$L_X = \frac{|Z_X|_{f_0}|}{2\pi f_0}, \text{ if } X \text{ is an inductor} \tag{9}$$

$$R_X = |Z_X|_{f_0}|, \text{ if } X \text{ is a resistor} \tag{10}$$

(d) **(10 pts)** Now we just need to calculate $H(\omega_0)$ using the samples of $v_{in}(t) = A_{in}\cos(2\pi f_0 t + \theta_{in})$ and $v_{out}(t) = A_{out}\cos(2\pi f_0 t + \theta_{out})$ that the on-board microcontroller collected. Suppose we have collected N samples from $v_{in}(t)$ and $v_{out}(t)$ with sampling interval $\Delta = \frac{1}{f_0 N}$, and we have arranged these samples into N-dimensional vectors $\vec{v}_{in}$ and $\vec{v}_{out}$. These sample vectors have the form

$$\vec{v}_{in} = \begin{bmatrix} A_{in}\cos(\theta_{in}) \\ A_{in}\cos\big(2\pi f_0(\Delta) + \theta_{in}\big) \\ A_{in}\cos\big(2\pi f_0(2\Delta) + \theta_{in}\big) \\ \vdots \\ A_{in}\cos\big(2\pi f_0((N-1)\Delta) + \theta_{in}\big) \end{bmatrix}, \quad \vec{v}_{out} = \begin{bmatrix} A_{out}\cos(\theta_{out}) \\ A_{out}\cos\big(2\pi f_0(\Delta) + \theta_{out}\big) \\ A_{out}\cos\big(2\pi f_0(2\Delta) + \theta_{out}\big) \\ \vdots \\ A_{out}\cos\big(2\pi f_0((N-1)\Delta) + \theta_{out}\big) \end{bmatrix}. \tag{11}$$

Since $\widetilde{V}_{out} = \frac{1}{2}A_{out}e^{j\theta_{out}}$ and $\widetilde{V}_{in} = \frac{1}{2}A_{in}e^{j\theta_{in}}$, these samples contain information about the currently unknown phasors $\widetilde{V}_{out}$ and $\widetilde{V}_{in}$ at the frequency f_0. These phasors need to be recovered from the samples using the DFT. **Express the DFTs $\vec{V}_{in} = F_N\vec{v}_{in}$ and $\vec{V}_{out} = F_N\vec{v}_{out}$ symbolically in terms of the phasors $\widetilde{V}_{out}$ and $\widetilde{V}_{in}$, the standard basis vectors $\vec{e}_i$, and the number of samples N.**

Here, the DFT transformation matrix $F_N = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N}1} & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}3} & \cdots & e^{-j\frac{2\pi}{N}(N-1)1} \\ 1 & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}4} & e^{-j\frac{2\pi}{N}6} & \cdots & e^{-j\frac{2\pi}{N}(N-1)2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi}{N}(N-1)} & e^{-j\frac{2\pi}{N}2(N-1)} & e^{-j\frac{2\pi}{N}3(N-1)} & \cdots & e^{-j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix}$

and the inverse of this matrix is just $\frac{1}{N}F_N^*$.

Solution: We'll show in great detail how to compute $\vec{V}_{out}$ here.

The exact same steps are applied to calculate $\vec{V}_{in}$: basically, you can take all of the steps you're about to see, and just replace "out" with "in".

We'll start by simplifying the sample vectors a bit. If we apply the value $\Delta = 1/(f_0 N)$ inside of the cosine terms, we have

$$\vec{v}_{out} = \begin{bmatrix} A_{out}\cos(\theta_{out}) \\ A_{out}\cos\left(\frac{2\pi}{N}1 + \theta_{out}\right) \\ A_{out}\cos\left(\frac{2\pi}{N}2 + \theta_{out}\right) \\ \vdots \\ A_{out}\cos\left(\frac{2\pi}{N}(N-1) + \theta_{out}\right) \end{bmatrix}. \tag{12}$$

Next, Euler's formula gives us $\cos(x) = \frac{1}{2}(e^{jx} + e^{-jx})$. If we apply this to all of the cosines in the sample vector, we get

$$\vec{v}_{out} = \begin{bmatrix} \frac{A_{out}}{2}\left(e^{j\theta_{out}} + e^{-j\theta_{out}}\right) \\ \frac{A_{out}}{2}\left(e^{j\left(\frac{2\pi}{N}1+\theta_{out}\right)} + e^{-j\left(\frac{2\pi}{N}2+\theta_{out}\right)}\right) \\ \frac{A_{out}}{2}\left(e^{j\left(\frac{2\pi}{N}1+\theta_{out}\right)} + e^{-j\left(\frac{2\pi}{N}2+\theta_{out}\right)}\right) \\ \vdots \\ \frac{A_{out}}{2}\left(e^{j\left(\frac{2\pi}{N}(N-1)+\theta_{out}\right)} + e^{-j\left(\frac{2\pi}{N}(N-1)+\theta_{out}\right)}\right) \end{bmatrix}. \tag{13}$$

Now, if we bring the common constants of each term outside of the vector, we get

$$\vec{v}_{out} = \frac{A_{out}}{2} e^{j\theta_{out}} \begin{bmatrix} 1 \\ e^{j\left(\frac{2\pi}{N}1\right)} \\ e^{j\left(\frac{2\pi}{N}2\right)} \\ \vdots \\ e^{j\left(\frac{2\pi}{N}(N-1)\right)} \end{bmatrix} + \frac{A_{out}}{2} e^{-j\theta_{out}} \begin{bmatrix} 1 \\ e^{-j\left(\frac{2\pi}{N}1\right)} \\ e^{-j\left(\frac{2\pi}{N}2\right)} \\ \vdots \\ e^{-j\left(\frac{2\pi}{N}(N-1)\right)} \end{bmatrix}. \tag{14}$$

We can recognize that the constants we just brought outside, namely $\frac{1}{2}A_{out}e^{j\theta_{out}}$ and $\frac{1}{2}A_{out}e^{-j\theta_{out}}$, are the phasor $\widetilde{V}_{out}$ and its complex conjugate $\overline{\widetilde{V}}_{out}$. Furthermore, we can recognize that the vectors are related to the columns of the DFT analysis matrix: spefically, we have

$$\vec{v}_{out} = \widetilde{V}_{out}\overline{\vec{u}}_1 + \overline{\widetilde{V}}_{out}\overline{\vec{u}}_{N-1}, \tag{15}$$

where $\vec{u}_i$ denotes the i^{th} column of the DFT analysis matrix F_N. With the sample vector written in this way, it's possible to compute its DFT by hand:

$$\vec{V}_{out} = F_N\vec{v}_{out} = \begin{bmatrix} \vec{u}_0^\top \\ \vec{u}_1^\top \\ \vdots \\ \vec{u}_{N-1}^\top \end{bmatrix} \left(\widetilde{V}_{out}\overline{\vec{u}}_1 + \overline{\widetilde{V}}_{out}\overline{\vec{u}}_{N-1}\right) = \widetilde{V}_{out} \begin{bmatrix} \vec{u}_0^\top\overline{\vec{u}}_1 \\ \vec{u}_1^\top\overline{\vec{u}}_1 \\ \vdots \\ \vec{u}_{N-1}^\top\overline{\vec{u}}_1 \end{bmatrix} + \overline{\widetilde{V}}_{out} \begin{bmatrix} \vec{u}_0^\top\overline{\vec{u}}_{N-1} \\ \vec{u}_1^\top\overline{\vec{u}}_{N-1} \\ \vdots \\ \vec{u}_{N-1}^\top\overline{\vec{u}}_{N-1} \end{bmatrix} \tag{16}$$

$$= \widetilde{V}_{out} \begin{bmatrix} 0 \\ N \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \overline{\widetilde{V}}_{out} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ N \end{bmatrix} = N\widetilde{V}_{out}\vec{e}_1 + N\overline{\widetilde{V}}_{out}\vec{e}_{N-1}. \tag{17}$$

By an analogous approach, we can do the same thing to the other sample vector $\vec{v}_{in}$. To summarize, we can express the DFTs of the two sample vectors as follows:

$$\vec{V}_{out} = N\widetilde{V}_{out}\vec{e}_1 + N\overline{\widetilde{V}}_{out}\vec{e}_{N-1} \tag{18}$$

$$\vec{V}_{in} = N\widetilde{V}_{in}\vec{e}_1 + N\overline{\widetilde{V}}_{in}\vec{e}_{N-1}. \tag{19}$$

5. DFT and Circuit Filters (30 pts)

You have been introduced to low-pass and high-pass filter circuits that pass some range of input signal frequencies while attenuating other ranges of signal frequency. You have also seen how we can break signals down and view the frequency components of sampled signals using the DFT. In this problem, we will see how we can combine these two bases of knowledge. Throughout this problem, if we have an N-dimensional vector $\vec{x}$, its DFT coefficients are given by the vector $\vec{X} = F_N\vec{x}$ where the DFT transformation matrix is

$$F_N = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N}1} & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}3} & \cdots & e^{-j\frac{2\pi}{N}(N-1)1} \\ 1 & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}4} & e^{-j\frac{2\pi}{N}6} & \cdots & e^{-j\frac{2\pi}{N}(N-1)2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi}{N}(N-1)} & e^{-j\frac{2\pi}{N}2(N-1)} & e^{-j\frac{2\pi}{N}3(N-1)} & \cdots & e^{-j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix}$$

and the inverse is $F_N^{-1} = \frac{1}{N}F_N^*$.

(a) **(6 pts)** If you sample every Δ seconds and you take N samples, the 0^{th} DFT coefficient $\vec{X}[0]$ corresponds to the DC (or constant) term. The 1^{st} DFT coefficient $\vec{X}[1]$ corresponds to the fundamental frequency $f_0 = \frac{1}{N\Delta}$.

Say you have a signal $v_{in}(t) = \cos\left(\frac{2\pi}{3}t\right) + \cos\left(\frac{2\pi}{9}t\right)$. You take $N = 9$ samples of the function every $\Delta = 1$ second; i.e. at $t = \{0, 1, 2, \ldots, 8\}$, forming a 9 element vector of samples $\vec{v}_{in}$. **What are the DFT coefficients $\vec{V}_{in}$ of the sampled signal $\vec{v}_{in}$?**

Solution:

We can rewrite

$$v_{in}(t) = \frac{1}{2}\left(e^{-j\frac{2\pi}{3}t} + e^{j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{9}t} + e^{j\frac{2\pi}{9}t}\right).$$

If we define the k^{th} row of the DFT matrix F_9 (with $N = 9$) to be

$$\vec{u}_k^T = \begin{bmatrix} e^{-j\frac{2\pi}{9}(k)(0)} & e^{-j\frac{2\pi}{9}(k)(1)} & \ldots & e^{-j\frac{2\pi}{9}(k)(8)} \end{bmatrix},$$

we can write the sampled version of $v_{in}(t)$ in vector notation as the column vector

$$\vec{v}_{in} = \frac{1}{2}\left(\vec{u}_3 + \overline{\vec{u}_3} + \vec{u}_1 + \overline{\vec{u}_1}\right).$$

One property of the rows of the DFT matrix is that $\vec{u}_k = \overline{\vec{u}_{N-k}}$, where N is the number of samples. Thus we can rewrite

$$\vec{v}_{in} = \frac{1}{2}(\vec{u}_3 + \vec{u}_6 + \vec{u}_1 + \vec{u}_8).$$

Since the rows of the DFT matrix are also orthogonal and have norm $\|\vec{u}_k\| = \sqrt{N}$, the inner product $\vec{u}_k^*\vec{u}_k = N$ and $\vec{u}_{i,\ i\neq k}^*\vec{u}_k = 0$. Therefore when we calculate

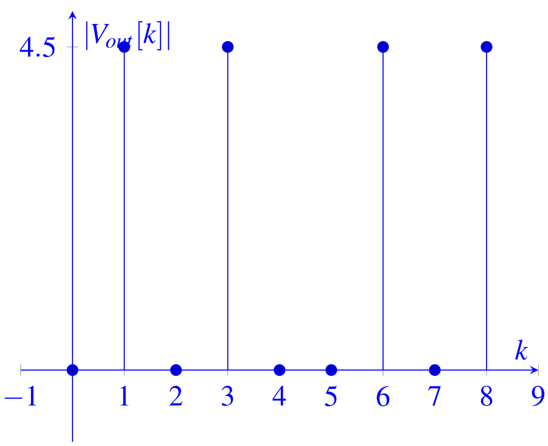
$$\vec{V}_{in} = F_9 \vec{v}_{in} = \frac{1}{2}\begin{bmatrix} \vec{u}_0^T \\ \vec{u}_1^T \\ \vec{u}_2^T \\ \vec{u}_3^T \\ \vec{u}_4^T \\ \vec{u}_5^T \\ \vec{u}_6^T \\ \vec{u}_7^T \\ \vec{u}_8^T \end{bmatrix} (\vec{u}_3 + \vec{u}_6 + \vec{u}_1 + \vec{u}_8) = \frac{1}{2}\begin{bmatrix} \vec{u}_0^* \\ \vec{u}_8^* \\ \vec{u}_7^* \\ \vec{u}_6^* \\ \vec{u}_5^* \\ \vec{u}_4^* \\ \vec{u}_3^* \\ \vec{u}_2^* \\ \vec{u}_1^* \end{bmatrix} (\vec{u}_3 + \vec{u}_6 + \vec{u}_1 + \vec{u}_8) = \frac{1}{2}\begin{bmatrix} 0 \\ 9 \\ 0 \\ 9 \\ 0 \\ 0 \\ 9 \\ 0 \\ 9 \end{bmatrix}$$

The coefficients are given by

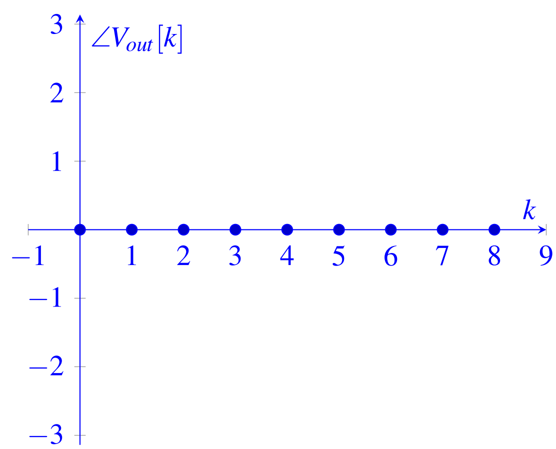
$$V_{out}[k] = \begin{cases} \frac{9}{2} & k = 1,3,6,8 \\ 0 & k = 0,2,4,5,7 \end{cases} \tag{20}$$

and can be plotted

Frequency Domain Magnitude



Frequency Domain Phase



(b) **(12 pts)** You are given the circuit below.

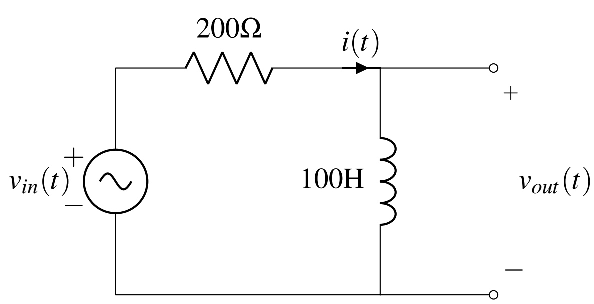


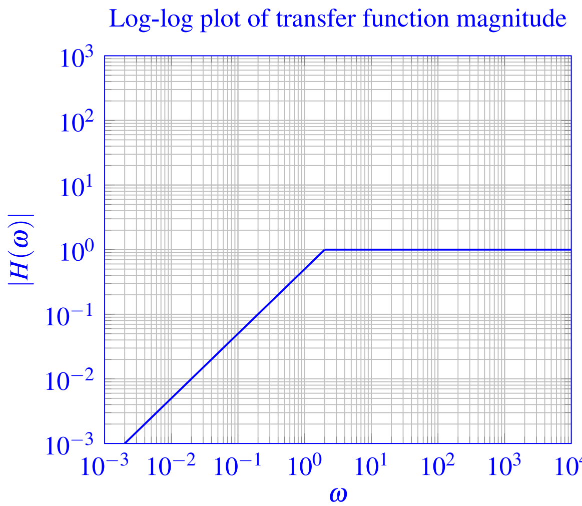
Figure 1: Filter circuit

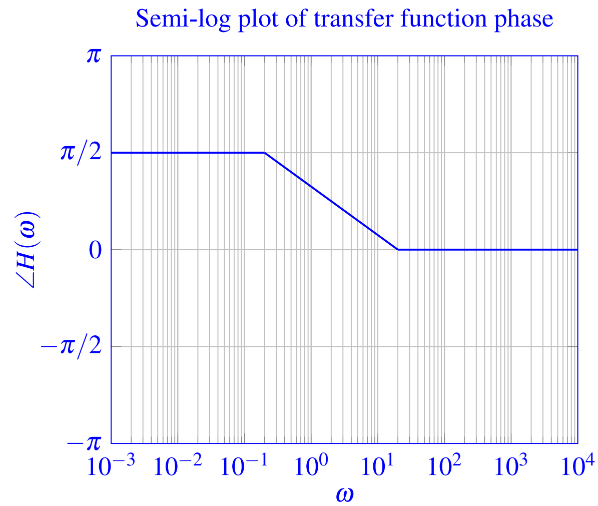
Is this a high-pass or low-pass filter? What is its cutoff angular frequency, ω_c? Sketch the piecewise-linear approximations of the magnitude and phase Bode plots of the transfer function $H(\omega) = \frac{\widetilde{V}_{out}(\omega)}{\widetilde{V}_{in}(\omega)}$ below.

Solution: Since the inductor will behave as a closed circuit for low-frequency and DC signals and as an open circuit for high-frequency signals, this is a high-pass filter.
From KVL, we know:

$$
\begin{aligned}
\widetilde{V}_{in} &= \widetilde{I}R + \widetilde{I}j\omega L \\
\widetilde{V}_{out} &= \widetilde{I}j\omega L \\
H(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} &= \frac{\widetilde{I}j\omega L}{\widetilde{I}R + \widetilde{I}j\omega L} = \frac{j\omega L}{R + j\omega L} = \frac{j\omega/(\frac{R}{L})}{1 + j\omega/(\frac{R}{L})}
\end{aligned}
$$

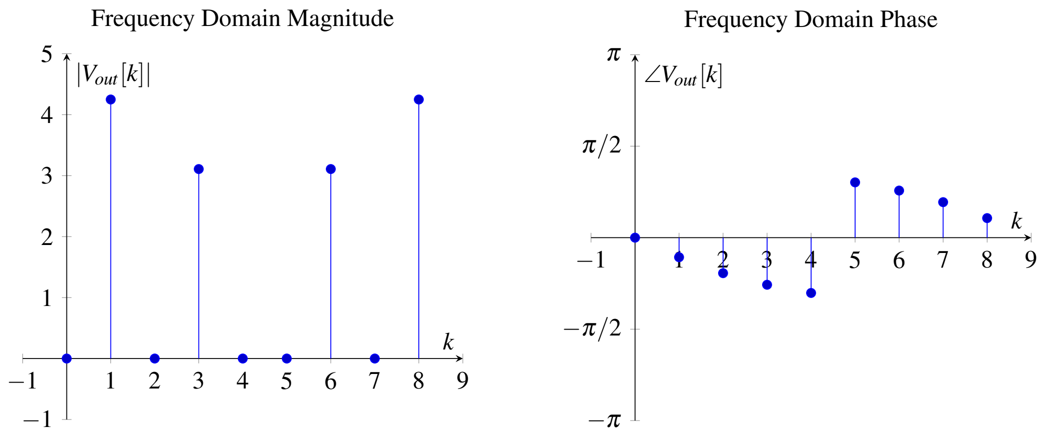
The cut-off frequency of the filter is $\omega_c = \frac{R}{L} = 2$ rad/sec $= 2 \times 10^0$ rad/sec.
We can draw the straight-line Bode plots by considering the behavior of the circuit for $\omega \to 0$ and $\omega \to \infty$. For $\omega \to 0$, $H(\omega \to 0) \approx \frac{j0}{1}$, giving $|H(\omega \to 0)| \approx 0$ and $\angle H(\omega \to 0) \approx \frac{\pi}{2}$ rad. For $\omega \to \infty$, $H(\omega \to \infty) \approx \frac{j\infty}{j\infty} \approx \frac{\infty}{\infty}$, giving $|H(\omega \to \infty)| \approx 1$ and $\angle H(\omega \to \infty) \approx 0$ rad.
The magnitude plot's "corner" occurs at ω_c. The angle plot's two "corners" occur at $0.1\omega_c$ and $10\omega_c$. The plots are shown below.



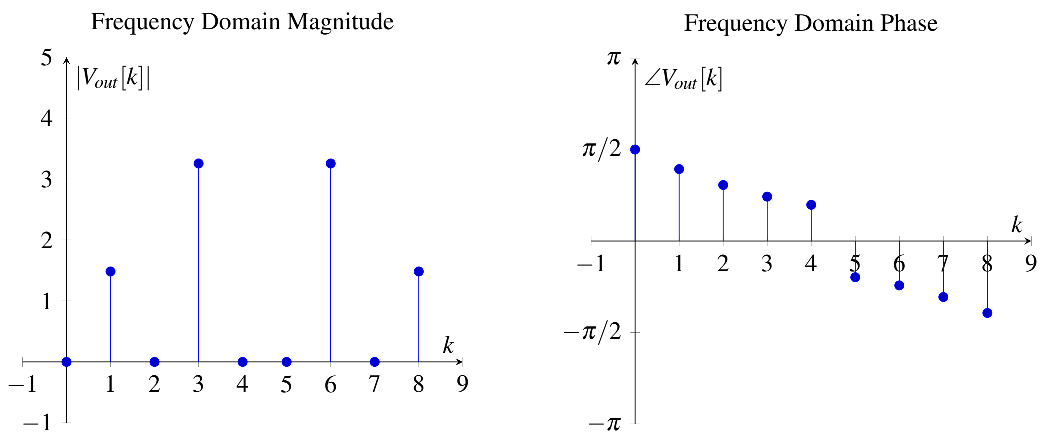


(c) **(12 pts)** The signal $v_{in}(t) = \cos\left(\frac{2\pi}{3}t\right) + \cos\left(\frac{2\pi}{9}t\right)$ is input into the circuit in Figure 1, giving output signal $v_{out}(t)$. You take $N = 9$ samples of the function $v_{out}(t)$ every $\Delta = 1$ seconds; i.e. at $t = \{0, 1, 2, \ldots, 8\}$, forming a 9 element vector of samples $\vec{v}_{out}$. We have given you several possible plots below that may represent the DFT coefficients $\vec{V}_{out}$ of the sampled signal $\vec{v}_{out}$. **For each of the four candidate solutions, circle the statement which is true. Provide a one-sentence explanation for your choice in the box provided.** *Reminder:* $\omega = 2\pi f$.

(HINT: Exactly one of the candidate solutions below is correct. Consequently, no precise numerical calculations are required to get full credit.)

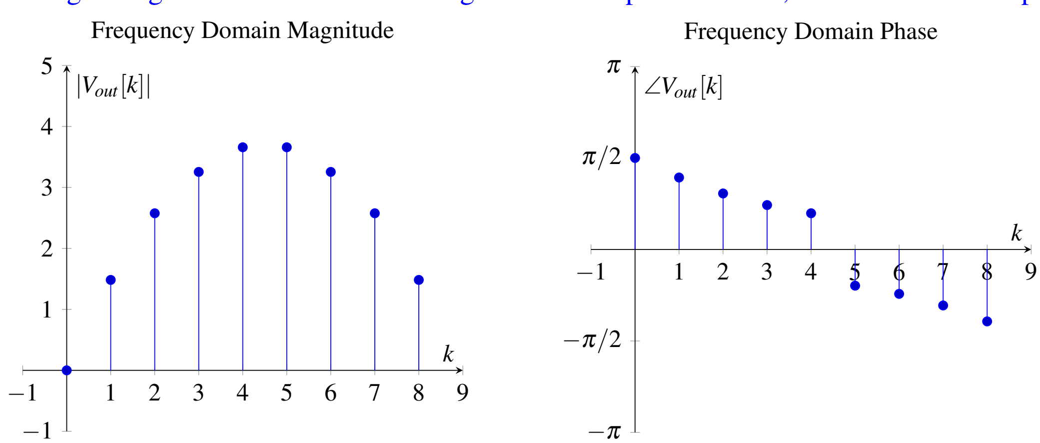


Solution: Incorrect. This is a low-pass filtered signal (not a high-pass filtered signal, as this circuit would produce).

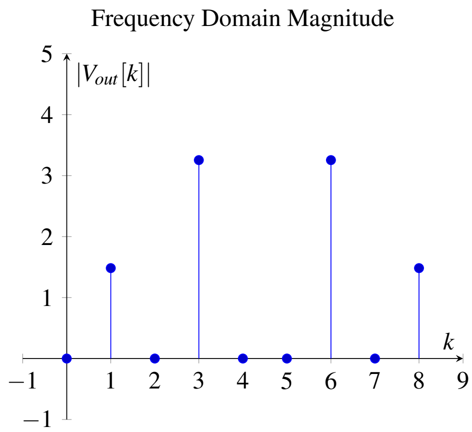


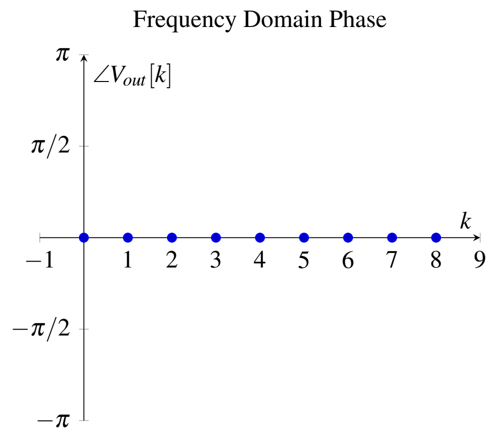
Solution: Correct. This is a high-pass filtered signal with the correct nonzero frequency components and correct phase. The signal is also conjugate-symmetric, as we would expect from a real signal.

Notice that here, we have phases that are not zero in the plot for magnitudes that are zero. There is nothing wrong with that since a zero magnitude corresponds to zero, no matter what the phase is.



Solution: Incorrect. Frequency components that were zero in the input will not increase in magnitude by being filtered.





Solution: Incorrect. The high-pass filtered magnitude is correct, but there is no phase change from the input, which is incorrect.

6. Lagrange Polynomials (24 pts)

In this question, we consider the interpolation of a function $f(x)$, at N points $x_0, \ldots, x_{N-1}$. The samples are collected in vector $\vec{f}_N = [f(x_0), f(x_1), \ldots, f(x_{N-1})]^\top$. The k-th component of $\vec{f}_N$ is denoted by $f_N[k]$.

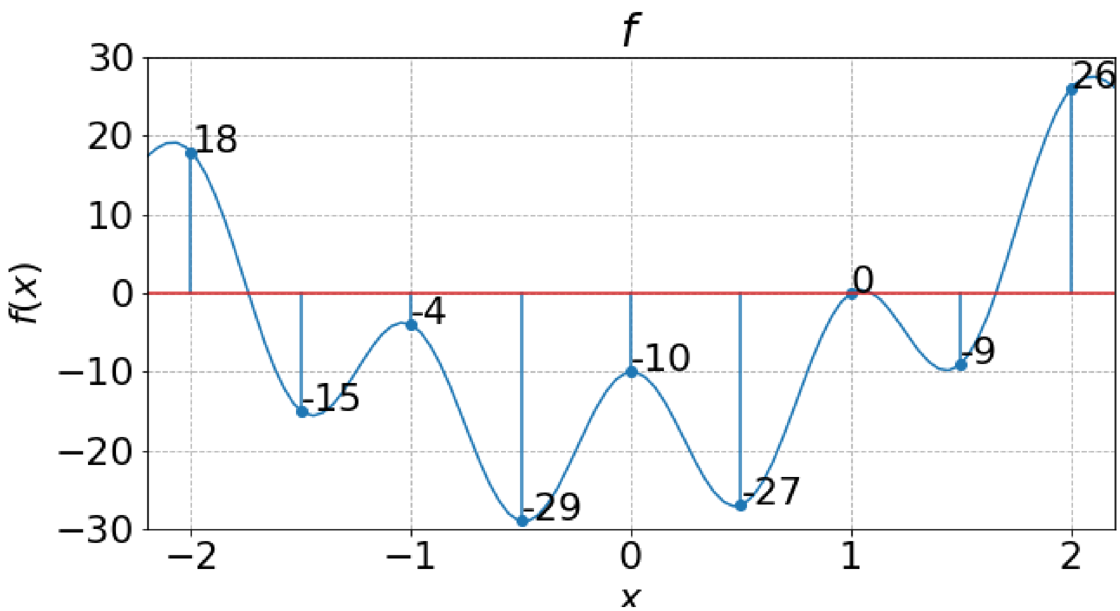


Figure 2: Plot of an example function $f(x)$

(a) **(2 pts)** For function $f(x)$ given in Figure 2, **give the vector $\vec{f}_8$ of samples for $x_k = -2 + \frac{k}{2}, k = 0, \ldots, 7$.**

Solution:

$$\vec{f}_8 = \begin{pmatrix} 18 & -15 & -4 & -29 & -10 & -27 & 0 & -9 \end{pmatrix}^\top$$

We just read off the relevant 8 values from the plot.

(b) **(6 pts)** Recall the family of Lagrange polynomials $\{L_i\}$ of degree at most $N-1$ from discussion and homework. For all $i = 0, \ldots, N-1$, the polynomial L_i is of degree at most $N-1$ and is given by:

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{N-1} \left(\frac{x - x_j}{x_i - x_j} \right) \tag{21}$$

Explicitly write out the vector $\vec{\ell}_i = [L_i(x_0), L_i(x_1), \ldots, L_i(x_{N-1})]^\top$ of samples for the i-th Lagrange polynomial L_i sampled at $x_0, x_1, \ldots, x_i, \ldots, x_{N-1}$, and argue why this family of vectors $\{\vec{\ell}_i\}$ is orthonormal.

Solution: For $i \in 0, \ldots, N-1$, by definition of L_i, we can see that $L_i(x_j) = 0$ for $j \neq i$ since the product will have a $(x_j - x_j)$ term in the numerator. All that remains is to understand what happens at $L_i(x_i) = \prod_{\substack{j=0 \\ j \neq i}}^{N-1} \left(\frac{x_i - x_j}{x_i - x_j} \right) = 1$ since every term in the product is 1. Putting this together, we have that $\vec{\ell}_i = \vec{e}_i$ where $\vec{e}_i$ is the standard i-th basis element with a 1 in the i-th position and zeros everywhere else.

The standard basis family is orthonormal since putting the $\{\vec{\ell}_i\}$ together into a matrix just gives the identity matrix, which is an orthonormal matrix.

Explicitly, $\vec{\ell}_i^T \vec{\ell}_j = 0$ if $i \neq j$ since they have nonzero elements in different positions. $\vec{\ell}_i^T \vec{\ell}_i = 1$ since $\|\vec{e}_i\| = 1$.

(c) **(4 pts)** For a sample vector $\vec{f}_N$, the polynomial H that interpolates it can be written $H(x) = \sum_{i=0}^{N-1} b[i]L_i(x)$. **Write** $b[i]$ **in terms of** $\vec{f}_N$, using the special properties of the Lagrange polynomials that you found in the previous part.

(Hint: Interpolation means that $H(x_j) = f_N[j]$ *for* $j = 0, \ldots, N-1$.*)*

Solution:

The answer here can be seen immediately to be For $i \in 0, \ldots, N-1$

$$b_i = \vec{f}_N[i] \tag{22}$$

But working this out, we see that we get a system of equations from $\vec{f}_N[j] = H(x_j) = \sum_{i=0}^{N-1} b[i]L_i(x_j)$. But every $L_i(x_j)$ term is zero except for $L_i(x_i)$ and so this interpolation equation becomes $\vec{f}_N[j] = b_j L_j(x_j) = b_j$, giving us the answer above.

(d) **(8 pts)** The same polynomial H can also be written as follows: For all x, $H(x) = \sum_{j=0}^{N-1} a[j]x^j$. For $j = 0, \ldots, N-1$, **write** $a[j]$ **in terms of** $\vec{f}_N$ **using the matrix form** $\vec{a} = P\vec{f}_N$**. What is the matrix** P **here?**

It is fine to leave your result for P in terms of other matrices and matrix operations, as long as it is explicit and unambiguous.

Solution: There are lots of ways of getting this. The direct path is just to write out the interpolation equations to get $\vec{f}_N[i] = H(x_i) = \sum_{j=0}^{N-1} a[j]x_i^j$ which can be expressed in matrix form as $\vec{f}_N = V\vec{a}$ where $V = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{N-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N-1} & x_{N-1}^2 & \cdots & x_{N-1}^{N-1} \end{bmatrix}$ is the Vandermonde matrix. (Each row corresponds to one of the interpolation equations.) Then, $\vec{a} = V^{-1}\vec{f}_N$ and so $P = V^{-1}$.

(e) **(4 pts)** If the samples are taken at $x_k = \omega_N^k = e^{j\frac{2\pi}{N}k}$, **do you recognize the** P **matrix that gives us the coefficients of the interpolating polynomial? What is it?**

Solution: With samples taken at ω_N^k for $k = 0, \ldots, N-1$, the Vandermonde matrix V would be:

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \cdots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \cdots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \cdots & \omega_N^{(N-1)(N-1)} \end{pmatrix} \tag{23}$$

This is because $\omega_N^0 = 1$.

Then from the previous question, $P = V^{-1}$. We recognize V as F_N^*. But remember that $NF_N^{-1} = F_N^*$, so $P = \frac{1}{N}F_N$. It is the standard (unnormalized) DFT matrix normalized by dividing through by N. This normalization makes sense if you think about it. This says that the estimate for the constant term should be the average of samples instead of just their sum.

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

7. Observability Lost (10 pts)

In this problem, we will be considering discrete-time systems with outputs, that is systems of the form

$$\vec{x}_d(t+1) = A\vec{x}_d(t) + B\vec{u}_d(t) \tag{24}$$
$$y_d(t) = C\vec{x}_d(t). \tag{25}$$

As you know, the overwhelming majority of such systems are observable. Nevertheless, there are some instances where systems that are not observable *do* arise.

Suppose that at least one of the eigenvectors of A is in the nullspace of C. That is, there exists at least one vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$ and $C\vec{v} = \vec{0}$. **Prove that, under these conditions, the system cannot be observable**.

Here, feel free to assume that $C = c^T$ is a row vector and that $y_d(t)$ is a scalar valued function of time.

Solution: Since this is a proof-type question, there are multiple distinct approaches to the problem that are valid. For the solution, we will show three such distinct approaches.

- Explicitly show that the observability matrix is rank-deficient by demonstrating that its nullspace contains a nonzero vector.

  Recall that a system with (A, C) is not observable if the matrix

  $$\mathscr{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{26}$$

  does not have full column rank, which we could prove is the case if we could find a vector $\vec{v}$ such that $\mathscr{O}\vec{v} = \vec{0}$.

  Since we know that at least one of the eigenvectors of A is in the nullspace of C, we know that there is a vector $\vec{v}$ (the eigenvector in question) such that $A\vec{v} = \lambda\vec{v}$ and $C\vec{v} = \vec{0}$. This turns out to be exactly the vector we need to prove that $\mathscr{O}$ doesn't have full column rank, since

  $$\mathscr{O}\vec{v} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\vec{v} = \begin{bmatrix} C\vec{v} \\ CA\vec{v} \\ \vdots \\ CA^{n-1}\vec{v} \end{bmatrix} = \begin{bmatrix} C\vec{v} \\ \lambda C\vec{v} \\ \vdots \\ \lambda^{n-1}C\vec{v} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{0} \\ \vdots \\ \vec{0} \end{bmatrix} = \vec{0}. \tag{27}$$

- Show that the observability matrix is rank-deficient by inspection in the eigenvector basis.

  Suppose that A has a n distinct eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$, so that the eigenvectors of A for a basis $\mathbb{R}^n$. Then we can diagonalize A, that is write $A = V\Lambda V^{-1}$, where V is the matrix of eigenvectors, and Λ is the diagonal matrix containing the corresponding eigenvalues. Also, for convenience, let the given eigenvector (the one we know is in the nullspace of C) be be $\vec{v}_1$, since we can organize the eigenvectors however we please.

  Now, let us perform a change of coordinates by expressing A and C in the coordinates of the eigenvector basis. Let $\widetilde{A}$ and $\widetilde{C}$ denote the representations of A and C matrices, respectively, under the new coordinates. Of course, we'll have $\widetilde{A} = V^{-1}AV = \Lambda$. Similarly, in the new coordinates the C matrix becomes $\widetilde{C} = CV$. We know that the first element of $\widetilde{C}$ is zero, since

$$\widetilde{C} = CV = C\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n\end{bmatrix} = \begin{bmatrix}C\vec{v}_1 & C\vec{v}_2 & \dots & C\vec{v}_n\end{bmatrix} = \begin{bmatrix}\vec{0} & \vec{q}_2 & \dots & C\vec{q}_n\end{bmatrix}$$

where the q_i vectors are arbitrary vectors whose exact value is not important for this argument.
Since $\widetilde{A}$ is diagonal in this bases, we also have

$$\widetilde{C}\widetilde{A}^k = \widetilde{C}\Lambda^k = \begin{bmatrix}\vec{0} & \vec{q}_2 & \dots & C\vec{q}_n\end{bmatrix}\begin{bmatrix}\lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k\end{bmatrix} = \begin{bmatrix}\vec{0} & \lambda_2^k\vec{q}_2 & \dots & \lambda_n^k\vec{q}_n\end{bmatrix}.$$

Using this expression, we can write the observability matrix in the eigenvector basis as

$$\widetilde{\mathscr{O}} = \begin{bmatrix}\widetilde{C} \\ \widetilde{C}\widetilde{A} \\ \vdots \\ \widetilde{C}\widetilde{A}^{n-1}\end{bmatrix} = \begin{bmatrix}\vec{0} & \vec{q}_2 & \dots & \vec{q}_n \\ \vec{0} & \lambda_2\vec{q}_2 & \dots & \lambda_n\vec{q}_n \\ \vdots & \vdots & \vdots & \vdots \\ \vec{0} & \lambda_2^{n-1}\vec{q}_2 & \dots & \lambda_n^{n-1}\vec{q}_n\end{bmatrix}.$$

In the eigenvector coordinates, the observability matrix $\widetilde{\mathscr{O}}$ does not have full column rank by inspection, since one of its columns is all zeros. This implies that the observability matrix in the original coordinates also does not have full column rank, meaning that the system is not observable.

While this is a valid argument, it assumes that the matrix A is diagonalizable, which was not a given assumption in the problem statement.

- Demonstrate that several initial conditions lead to the same sequence of outputs, meaning that observing a specific $y(t)$ does not have "enough information" to conclude a unique initial condition.

  This argument circumvents the usual rank test for observability, and shows directly that the definition of observability can be violated under the conditions given in the problem.

  Recall that for a system to be observable, it must be true that a given sequence of inputs $\vec{u}(t)$ and outputs $\vec{y}(t)$ must uniquely specify an initial condition $\vec{x}_0$. In other words, if there are multiple initial conditions that lead to the same outputs $\vec{y}(t)$ for a fixed input $\vec{u}(t)$, then the system is not observable.

  Suppose that we choose $\vec{u}(t) = \vec{0}$. Then, the system output will be

$$\vec{y}(t) = CA^t\vec{x}_0.$$

  Now, let $\vec{x}_0 = \alpha\vec{v}$, where $\alpha \in \mathbb{R}$ is an arbitrary scaling factor. No matter what value we let α take, the output will be

$$\vec{y}(t) = CA^t(\alpha\vec{v}) = \alpha C(A^t\vec{v}) = \alpha\lambda^t C\vec{v} = \vec{0}.$$

  This means that an infinite number of initial conditions all lead to the output $\vec{y}(t) = \vec{0}$ for the fixed input $\vec{u}(t) = 0$. Therefore, the system is not observable.

8. Real Eigenvalues (15 pts)

Suppose S is a complex matrix that can be written in the form $S = B^*B$, for some other complex matrix B. **Show that the eigenvalues of S are all real and non-negative.**

(Hint: Remember that $\vec{v}^*\vec{v} = \|\vec{v}\|^2 \geq 0$ for all $\vec{v}$ and that the norm $\|\cdot\|$ is always real valued, even for complex vectors.)

Solution: Consider an eigenvalue λ of S, with eigenvector $\vec{p}$.

$$S\vec{p} = \lambda\vec{p}$$

$$\implies B^*B\vec{p} = \lambda\vec{p}$$

If we pre-multiply both sides of the above equation by $\vec{p}^*$, then since $\vec{p}^*\vec{p} = ||\vec{p}||^2$, we get:

$$\vec{p}^*B^*B\vec{p} = \lambda||\vec{p}||^2$$

Now, $\vec{p}^*B^* = (B\vec{p})^*$,

$$\implies (B\vec{p})^*(B\vec{p}) = \lambda||\vec{p}||^2$$

Further, $(B\vec{p})^*(B\vec{p}) = ||B\vec{p}||^2$,

$$\implies ||B\vec{p}||^2 = \lambda||\vec{p}||^2$$

$$\therefore \lambda = \frac{||B\vec{p}||^2}{||\vec{p}||^2}$$

Now, since the right hand side of this equation is always real and non-negative, so is λ (where λ could be any eigenvalue of S).

PRINT your name and student ID: ____________________

9. Weighted minimum norm (25 pts)

You saw in lecture in the context of open-loop control, how we consider problems in which we have a wide matrix A and solve $A\vec{x} = \vec{y}$ such that $\vec{x}$ is a minimum norm solution:

$$\|\vec{x}\| \leq \|\vec{z}\|$$

for all $\vec{z}$ such that $A\vec{z} = \vec{y}$. You then saw this idea again in the homeworks in the context of MIMO communication and also worked out how to compute the appropriate "pseudo-inverse" for such wide matrices.

But what if you weren't interested in just the norm of $\vec{x}$? What if you instead cared about minimizing the norm of a linear transformation $C\vec{x}$? For example, suppose that controls were more or less costly at different times.

The problem can be written out mathematically as:

Given a wide matrix A and a matrix C find $\vec{x}$ such that $A\vec{x} = \vec{y}$ and $\|C\vec{x}\| \leq \|C\vec{z}\|$ for all $\vec{z}$ such that $A\vec{z} = \vec{y}$.

(a) **(10 pts)** Let's start with the case of C being invertible. **Solve this problem (i.e. find the optimal $\vec{x}$ with the minimum $\|C\vec{x}\|$) for the specific matrices and $\vec{y}$ given below. Show your work.**
It is fine to leave your answer as an explicit product of matrices and vectors.
(HINT: You might want to change variables to solve this problem. Don't forget to change back!)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For convenience, $C^{-1} = \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0 \end{bmatrix}$ and you are also given some SVDs on the following page.

$$A = \left(U_A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\left(\Sigma_A = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right)\left(V_A^T = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\right) \tag{28}$$

$$C = \left(U_C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\right)\left(\Sigma_C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\left(V_C^T = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) \tag{29}$$

$$AC = \left(U_{AC} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\left(\Sigma_{AC} = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\right)\left(V_{AC}^T = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix}\right) \tag{30}$$

$$AC^{-1} = \left(U_{AC^{-1}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\left(\Sigma_{AC^{-1}} = \begin{bmatrix} \frac{\sqrt{5}}{2} & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}\right)\left(V_{AC^{-1}}^T = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{bmatrix}\right) \tag{31}$$

Solution: In homework and lecture, you solved a similar problem $A\vec{x} = \vec{y}$ such that $\vec{x}$ is a minimum norm solution: $\|\vec{x}\| \leq \|\vec{z}\|$ for any $\vec{z}$ that satisfies $A\vec{z} = \vec{y}$.

When you solved this problem, you computed the appropriate psuedoinverse to solve for $\vec{x}$. This was the Moore Penrose psuedo inverse — sometimes depicted as $A^{\dagger}$.

Seeing that we already know how to solve such problems, we can first try to reformulate the current problem: $A\vec{x} = \vec{y}$ such that $\|C\vec{x}\| \le \|C\vec{z}\|$, into the problem that we already know how to solve. To do this we can do a change of variables (as the hint told us to do). Using the change of variables $\widetilde{x} = C\vec{x}$ and $\vec{p} = C\vec{z}$ we get the new constraint: $\|\widetilde{x}\| \le \|\vec{p}\|$ for any vector $\vec{p}$ that satisfies something. What is this something?

Originally, we had $A\vec{x} = \vec{y}$ and so in the changed variables, we have $\vec{x} = C^{-1}\widetilde{x}$ and so the constraint that needs to be satisfied is $AC^{-1}\widetilde{x} = \vec{y}$.

So our new problem is to solve $AC^{-1}\widetilde{x} = \vec{y}$ such that $\widetilde{x}$ is a minimum norm solution: $\|\widetilde{x}\| \le \|\vec{p}\|$ for all $\vec{p}$ that satisfy $AC^{-1}\vec{p} = \vec{y}$.

This is exactly like the minimum norm question on the homework except now the matrix multiplying the vector is AC^{-1}.

To solve this we proceed exactly like we did in the homework and find the Moore Penrose psuedo inverse of AC^{-1}:

$$\widetilde{x} = V_{compact,AC^{-1}} \Sigma^{-1}_{compact,AC^{-1}} U^T_{AC^{-1}} \vec{y} \tag{32}$$

where here, we need to be using the compact form of the SVD vis-a-vis AC^{-1}. Why compact? We need the Σ matrix to be square so we can invert it. This just means that we drop the parts of V^T that are just a basis for the nullspace of AC^{-1} — the last row. To be explicit, the compact SVD is:

$$AC^{-1} = (U_{AC^{-1}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})(\Sigma_{compact,AC^{-1}} = \begin{bmatrix} \frac{\sqrt{5}}{2} & 0 \\ 0 & 0.5 \end{bmatrix})(V^T_{compact,AC^{-1}} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix}). \tag{33}$$

Calculating this out, we get that $\widetilde{x} = \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \\ 4 \end{bmatrix}$.

However since the original question was to find $\vec{x}$ we have one more substitution to arrive at our final answer:

$$\vec{x} = C^{-1}\widetilde{x} = C^{-1} V_{compact,AC^{-1}} \Sigma^{-1}_{compact,AC^{-1}} U^T_{AC^{-1}} \vec{y} \tag{34}$$

$$= \begin{bmatrix} 2 \\ 0.8 \\ 0.2 \end{bmatrix}. \tag{35}$$

This could also have been solved (for full credit) by brute-force using calculus in this particular case since the first coordinate is forced to be 2 in order to get the desired first coordinate of $\vec{y}$. So we have only two variables left, with one linear constraint, which means that we could reduce the problem to one of minization of a quadratic function in one variable. However, this brute force calculus-based approach doesn't help us deal with the next part of this problem.

(b) **(15 pts)** What if C were a tall matrix with linearly independent columns? **Explicitly describe how you would solve this problem in that case, step by step.**

For convenience, we have copied the problem statement again here: Given a wide matrix A and a matrix C **find** $\vec{x}$ such that $A\vec{x} = \vec{y}$ and $\|C\vec{x}\| \le \|C\vec{z}\|$ for all $\vec{z}$ such that $A\vec{z} = \vec{y}$.

Here, you can assume that the wide matrix A has linearly-independent rows but is otherwise generic. Similarly, $\vec{y}$ is a generic vector.

(HINT: Does C have a nullspace? Does $C^T C$ have a nullspace? Does the SVD of C suggest any (invertible) change of coordinates from $\vec{x}$ to $\vec{\tilde{x}}$ such that $\|\vec{\tilde{x}}\| = \|C\vec{x}\|$?)

Solution:

Now we have the condition where C is a tall matrix with linearly independent columns. This means that C itself is no longer invertible and we cannot just repeat the procedure done in the previous part of the problem. We don't have access to a C^{-1} and so need to stop and think. What we want is a square matrix $\widetilde{C}$ that is invertible, and gives us the same norm to minimize. That is, we need $\|\widetilde{C}\vec{x}\| = \|C\vec{x}\|$.

Writing this out, we see that since $\|C\vec{x}\|^2 = \vec{x}^T C^T C\vec{x}$, what we want is that $C^T C = \widetilde{C}^T \widetilde{C}$. Following the hint and using the compact-form SVD of $C = U_{compact,C}\Sigma_{compact,C}V_C^T$ in which $\Sigma_{compact,C}$ is square. So, $C^T C = V_C \Sigma^2_{compact,C} V_C^T$ since $U_{compact,C}$ has orthonormal columns. This immediately suggests using $\widetilde{C} = \Sigma_{compact,C}V_C^T$. Clearly $C^T C = \widetilde{C}^T \widetilde{C}$ by construction.

The only question now is whether $\widetilde{C}$ is invertible. Because C has linearly independent columns, it cannot have a nullspace. But we know from lecture that if $C^T C\vec{v} = \vec{0}$, that indeed $C\vec{v} = \vec{0}$ and so $C^T C$ also does not have a nullspace. So $C^T C$ is invertible, and since $V_C \Sigma^2_{compact,C} V_C^T$ is the diagonalization of $C^T C$ by the basis V_C of eigenvectors, $\Sigma_{compact,C}$ is also invertible. The product of invertible matrices is invertible, and so indeed $\widetilde{C}$ is invertible.

At this point, we have reduced this problem to what we did in the previous part. We just want to minimize $\|\widetilde{C}\vec{x}\|$ over all $\vec{x}$ that satisfy $A\vec{x} = \vec{y}$. This is equivalent to minimizing $\|\vec{\tilde{x}}\|$ over all $\vec{\tilde{x}}$ that satisfy $A\widetilde{C}^{-1}\vec{\tilde{x}} = \vec{y}$.

So in terms of an explicit procedure:

i. Compute the compact SVD of $C = U_{compact,C}\Sigma_{compact,C}V_C^T$.

ii. Compute the matrix $\widetilde{C} = \Sigma_{compact,C}V_C^T$.

iii. Compute the compact form SVD of the matrix $A\widetilde{C}^{-1} = U\Sigma V^T$.

iv. Compute the solution $\vec{x} = \widetilde{C}^{-1} V \Sigma^{-1} U^T \vec{y}$.

This comes from changing variables to $\vec{x} = \widetilde{C}^{-1}\vec{\tilde{x}}$ and finding the minimum norm $\vec{\tilde{x}}$ that works.

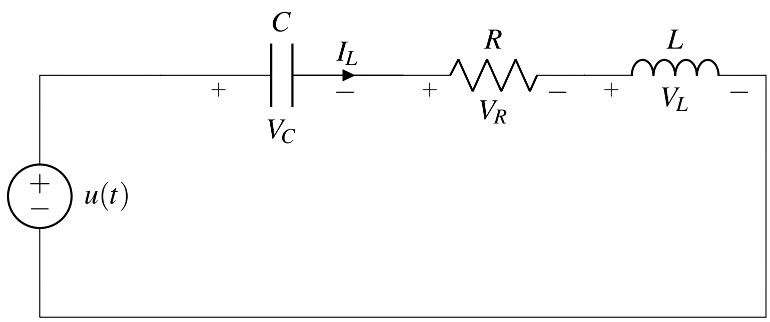
An alternative solution (that amounts to the same thing, effectively) exists where we use the pseudo-inverse of C (i.e. the least-squares solution) and build our solution around that instead. Arguing why that works is a bit more involved.

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

PRINT your name and student ID: ____

10. Circuit Discretization (18 pts)

Let's consider the following RLC circuit that you have encountered before.



(a) **(6 pts)** Find the matrix differential equation for the above system using the state-vector $\vec{x} = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix}$ as

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t).$$

What is A? What is $\vec{b}$?

Your answers should be in terms of R, L, C.

Solution: Writing the circuit equations, we get:

$$\begin{aligned} u(t) &= V_C + V_R + V_L \\ V_L &= L\frac{d}{dt}I_L \\ V_R &= I_L R \\ I_L &= C\frac{d}{dt}V_C \end{aligned}$$

Substituting the defintions, we get:

$$\begin{aligned} u(t) &= V_C + I_L R + L\frac{d}{dt}I_L \\ \Rightarrow \frac{d}{dt}I_L &= -\frac{1}{L}V_C - \frac{R}{L}I_L + \frac{1}{L}u(t) \\ \frac{d}{dt}V_C &= \frac{1}{C}I_L \end{aligned}$$

Hence, we can write the matrix differential equation as

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}u(t)$$

(b) **(12 pts)** Now, assume for some specific component values we get the following differential equation:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix}u(t). \tag{36}$$

Unfortunately, we are unable to measure our state vector continuously. Suppose that we sample the system with some sampling interval Δ. Let us discretize the above system. Assume that we use piecewise constant voltage inputs $u(t) = u_d(k)$ for $t \in [k\Delta, (k+1)\Delta)$.

Recall from the homework that for a hypothetical scalar differential equation $\frac{d}{dt}x(t) = \lambda x(t) + bu(t)$, we can discretize it as long as $\lambda \neq 0$ as follows:

$$x_d(k+1) = e^{\lambda\Delta}x_d(k) + \frac{e^{\lambda\Delta}-1}{\lambda}bu_d(k). \tag{37}$$

Here $x_d(k) = x(k\Delta)$.

Using equation (37), calculate the discrete-time system for Equation (36)'s continuous-time vector system in the form:

$$\vec{x}_d(k+1) = A_d\vec{x}_d(k) + \vec{b}_d u_d(k).$$

More concretely, find A_d and $\vec{b}_d$.

You do not need to multiply out any matrices. It is fine if you give your answers as explicit products of matrices/vectors/etc.

Hint: We have provided information regarding the matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ in (36) for your convenience (not all of this is needed) on the opposite page.

i. The determinant of A: $\det(A) = 2$.

ii. The trace of A: $\operatorname{tr}(A) = -3$.

iii. $A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}$.

iv. We can diagonalize the matrix as $A = V\Lambda V^{-1}$, where, Λ is a diagonal matrix with the eigenvalues in its diagonal and the columns of V are the eigenvectors of the corresponding eigenvalues

v. The eigenvalues/eigenvectors for A are:

$$\text{For } \lambda_1 = -2: \ \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{For } \lambda_2 = -1: \ \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

vi. For $V = [\vec{v}_1, \vec{v}_2]$, we have $V^{-1} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$.

Solution:

We want to change coordinates to the eigenbasis, so that the system of differential equations looks like scalar equations. Having done so, we can discretize the problem, and then change coordinates back. We can write $A = V\Lambda V^{-1}$, hence substituting this into our differential equation, we get:

$$\frac{d}{dt}\vec{x} = V\Lambda V^{-1}\vec{x} + \vec{b}u(t)$$
$$\frac{d}{dt}V^{-1}\vec{x} = \Lambda V^{-1}\vec{x} + V^{-1}\vec{b}u(t)$$

Writing, $\vec{z} = V^{-1}\vec{x}$, we can diagonlize the system. Hence, we can discretize the system in this diagonal space, giving us

$$\vec{z}_d(k+1) = e^{\Lambda\Delta}\vec{z}_d(k) + \Lambda_\Delta V^{-1}\vec{b}u(t).$$

Here, $\vec{z}_d(k) = \vec{z}(k\Delta)$ and $\Lambda_\Delta = \text{diag}(\frac{e^{\lambda_j\Delta}-1}{\lambda_j})$ – for the j-th entry of the diagonal. Fortunately, in our case, all the eigenvalues $\lambda_j \neq 0$ and so this applies. This is just applying the scalar solution we were given in the problem to each of the components of $\vec{z}$ in the discretization.

Hence, substituting back for $\vec{x}_d(k)$ gives

$$\begin{aligned} V^{-1}x_d(k+1) &= e^{\Lambda\Delta}V^{-1}\vec{x}_d(k) + \Lambda_\Delta V^{-1}\vec{b}u(t) \\ x_d(k+1) &= Ve^{\Lambda\Delta}V^{-1}\vec{x}_d(k) + V\Lambda_\Delta V^{-1}\vec{b}u(t). \end{aligned}$$

This gives:

$$\begin{aligned} A_d &= Ve^{\Lambda\Delta}V^{-1} \\ \vec{b}_d &= V\Lambda_\Delta V^{-1}\vec{b} \end{aligned}$$

Hence we have,

$$A_d = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{-2\Delta} & 0 \\ 0 & e^{-\Delta} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

$$\vec{b}_d = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{e^{-2\Delta}-1}{2} & 0 \\ 0 & -(e^{-\Delta}-1) \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Multiplying out the matrices, we get:

$$\begin{aligned} A_d &= \begin{bmatrix} 2e^{-\Delta} - e^{-2\Delta} & e^{-\Delta} - e^{-2\Delta} \\ 2e^{-2\Delta} - 2e^{-\Delta} & 2e^{-2\Delta} - e^{-\Delta} \end{bmatrix} \\ \vec{b}_d &= \begin{bmatrix} e^{-2\Delta} - 2e^{-\Delta} + 1 \\ 2e^{-\Delta} - 2e^{-2\Delta} \end{bmatrix} \end{aligned}$$

Note: You were not asked to multiply out the matrices in the exam.

11. Discretization With Piecewise Linear Controls (18 pts)

In most of this course, when discretizing a continuous-time control system, we forced the input to be constant between time steps, *i.e.*, between some $k\Delta$ and $(k+1)\Delta$, (this is alternatively called a zero-order hold) and then changed it instantly and discontinuously to its new value. However, applying such a discontinuous control might be physically impossible for a real-world system. Suppose we decided instead to use something piecewise-linear (see Figure 3) for our continuous-time input.

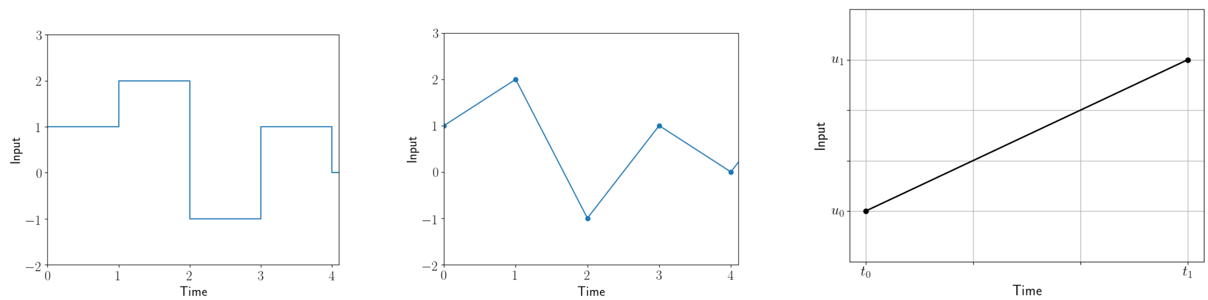


Figure 3: Piecewise constant vs. piecewise linear control inputs and a pure affine control input.

Consider a scalar differential-equation with scalar input $u(t)$:

$$\frac{d}{dt}x(t) = \lambda x(t) + bu(t) \tag{38}$$

with an initial condition $x(t_0)$. If we use a pure affine input (see the third panel in the figure above) for $u(t)$, we get the following continuous-time scalar differential equation:

$$\frac{d}{dt}x(t) = \lambda x(t) + b\left(m(t-t_0)+u_0\right) \tag{39}$$

where $m = \left(\frac{u_1-u_0}{t_1-t_0}\right)$ is the slope of the input $u(t)$ and u_0 is where the input $u(t)$ starts at time t_0, with u_1 being where the input $u(t)$ ends at time $t_1 > t_0$. Assuming $\lambda \neq 0$, solving this differential equation (39) for an arbitrary initial condition $x(t_0)$, we get the following solution for all $t_0 \leq t \leq t_1$:

$$x(t) = x(t_0)e^{\lambda(t-t_0)} - \frac{b}{\lambda}m(t-t_0) + \frac{b}{\lambda}\left(\frac{m}{\lambda}+u_0\right)(e^{\lambda(t-t_0)}-1). \tag{40}$$

The goal in this problem is to extend (40) to let us discretize the continuous-time differential equation (38) under piecewise-linear inputs. The twist comes from the fact that each linear segment is defined by two numbers — "slope" and "intercept."

(a) **(6 pts)** The first step in discretizing Equation (38) is to consider each discrete time step (between $t = k\Delta$ and $t = (k+1)\Delta$) as virtually giving us not one, but two discrete-time inputs $\begin{bmatrix} s_d(k) \\ m_d(k) \end{bmatrix}$. Namely: $s_d(k) = u(k\Delta)$, the "intercept" where the input $u(t)$ starts for this interval and $m_d(k) = \frac{u((k+1)\Delta)-u(k\Delta)}{\Delta}$ as the "slope" of the $u(t)$ input in the interval between $t = k\Delta$ and $t = (k+1)\Delta$.

We can write the behavior of the discrete-time state $x_d(k) = x(k\Delta)$ as obeying a scalar discrete-time controlled recurrence relation:

$$x_d(k+1) = \lambda_d x_d(k) + b_{d,m} m_d(k) + b_{d,s} s_d(k). \tag{41}$$

What are λ_d, $b_{d,m}$, $b_{d,s}$ in terms of the given λ, Δ, b?

(Hint: use Equation (40) *as appropriate.)*

Solution: Since we are trying to discretize this system with a linearly interpolated input, we only care about the values of x at every Δ interval of time. Similar to the case of piecewise constant interpolation seen in HW8, the value at the previous time-step acts as the inital condition for the differential equation at the current time step. Hence, we have $t_0 = k\Delta$ and $t = (k+1)\Delta$. Furthermore, we have $u_1 = u((k+1)\Delta)$ and $u_0 = u(k\Delta)$.

Hence we have $\Delta = t - t_0$. Appropriately substituting the variables and grouping together terms, we get

$$x_d(k+1) = e^{\lambda\Delta} x_d(k) + \left(\frac{b}{\lambda^2}(e^{\lambda\Delta} - 1) - \frac{b\Delta}{\lambda}\right) m_d(k) + \frac{b}{\lambda}(e^{\lambda\Delta} - 1) s_d(k).$$

Hence, we can see that

$$\lambda_d = e^{\lambda\Delta}.$$

$$b_{d,m} = \left(\frac{b}{\lambda^2}(e^{\lambda\Delta} - 1) - \frac{b\Delta}{\lambda}\right).$$

$$b_{d,s} = \frac{b}{\lambda}(e^{\lambda\Delta} - 1).$$

(b) **(4 pts)** We want to understand how this system behaves in discrete-time as a function of the sequence of endpoints $u_d(k) = u((k+1)\Delta)$ of the piecewise constant input $u(t)$.

In reality, both the $m_d(k)$ input and the $s_d(k)$ input depend on $u_d(k)$ and $u_d(k-1)$. **Find $b_{d,1}$ and $b_{d,2}$ (in terms of the $\Delta, \lambda_d, b_{d,m}, b_{d,s}$ from above) so that Equation** (41) **can be rewritten as:**

$$x_d(k+1) = \lambda_d x_d(k) + b_{d,1} u_d(k) + b_{d,2} u_d(k-1). \tag{42}$$

(HINT: remember how $m_d(k)$ was defined. And that $u_d(k)$ is the $u(t)$ at the end of the interval and $u_d(k-1)$ is the $u(t)$ at the beginning of the interval from $[k\Delta, (k+1)\Delta]$.)

Solution: This question is just asking to rewrite the discrete time equation in terms of $u_d(k)$ and $u_d(k-1)$. Substituting for $m_d(k) = \frac{u_d(k) - u_d(k-1)}{\Delta}$ and grouping terms, we get

$$\begin{aligned} x_d(k+1) &= \lambda_d x_d(k) + \left(\frac{b_{d,m}}{\Delta}\right) u_d(k) - \left(\frac{b_{d,m}}{\Delta} - b_{d,s}\right) u_d(k-1) \\ &= \lambda_d x_d(k) + \frac{b}{\lambda\Delta}\left(\frac{1}{\lambda}(e^{\lambda\Delta} - 1) - \Delta\right) u_d(k) - \frac{b}{\lambda}\left[\left(\frac{1}{\lambda\Delta} - 1\right)(e^{\lambda\Delta} - 1) - 1\right] u_d(k-1) \end{aligned}$$

Hence, we can write

$$\begin{aligned}
\lambda_d &= e^{\lambda\Delta} \\
b_{d,1} &= \left(\frac{b_{d,m}}{\Delta}\right) \\
&= \frac{b}{\lambda\Delta}\left(\frac{1}{\lambda}(e^{\lambda\Delta}-1)-\Delta\right) \\
&\text{(OR)} \\
&= \frac{1}{\Delta}\left(-\frac{b\Delta}{\lambda}+\frac{b}{\lambda^2}(e^{\lambda\Delta}-1)\right) \\
b_{d,2} &= -\left(\frac{b_{d,m}}{\Delta}-b_{d,s}\right) \\
&= -\frac{b}{\lambda}\left[\left(\frac{1}{\lambda\Delta}-1\right)(e^{\lambda\Delta}-1)-1\right] \\
&\text{(OR)} \\
&= \frac{b}{\lambda}\left[1+(e^{\lambda\Delta}-1)\left(1-\frac{1}{\lambda\Delta}\right)\right]
\end{aligned}$$

(c) **(8 pts)** To get this into standard vector/matrix form, we realize that we need to remember $u_d(k-1)$ for the next time step. Everything that needs to be remembered has be a part of the state, and so let's augment our state vector as

$$\widetilde{\vec{x}}_d(k) = \begin{bmatrix} x_d(k) \\ u_d(k-1) \end{bmatrix}.$$

Starting with Equation (42), write a matrix time-evolution equation using $\widetilde{\vec{x}}_d$ as

$$\widetilde{\vec{x}}_d(k+1) = A_d\widetilde{\vec{x}}_d(k) + \vec{b}_d u_d(k).$$

More concretely, find A_d and $\vec{b}_d$, in terms of λ_d, $b_{d,1}$ and $b_{d,2}$.

Solution: From the previous part, we have

$$x_d(k+1) = \lambda_d x_d(k) + b_{d,1}u_d(k) + b_{d,2}u_d(k-1)$$

By using the given definitions, we can rewrite the above scalar equation as a matrix equation as follows:

$$\begin{aligned}
\widetilde{\vec{x}}(k+1) &= \begin{bmatrix} x_d(k+1) \\ u_d(k) \end{bmatrix} \\
&= \begin{bmatrix} \lambda_d & b_{d,2} \\ 0 & 0 \end{bmatrix}\widetilde{\vec{x}}(k) + \begin{bmatrix} b_{d,1} \\ 1 \end{bmatrix} u_d(k)
\end{aligned}$$

Hence, $\widetilde{A} = \begin{bmatrix} \lambda_d & b_{d,2} \\ 0 & 0 \end{bmatrix}$ and $\widetilde{\vec{b}} = \begin{bmatrix} b_{d,1} \\ 1 \end{bmatrix}$.

What this shows is that we can, in general, incorporate piecewise-linear controls into our discretization procedure, just at the cost of adding one more discrete-time state for every dimension of control input. This new extra state reflects state within our own controller — namely the memory within the system that piecewise-linearly interpolates the continuous-time control inputs between discrete-time setpoints.

PRINT your name and student ID: ____

[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.]

Contributors:

- Nikhil Shinde.
- Alex Devonport.
- Regina Eckert.
- Geoffrey Négiar.
- Sanjit Batra.
- Jaijeet Roychowdhury.
- Aditya Arun.
- Anant Sahai.