MATH 54 MIDTERM 1 – October 3 2019 5:10-6:30pm

Your Name Student ID	SOLUTIONS

Please exchange student IDs to record the

names of your two closest seat neighbors	

Do not turn this page until you are instructed to do so.

Show all your work in this exam booklet. There are blank pages for scratch work, but please do not remove any pages! **If you want something on an extra page to be graded, label it by the problem number and write "XTRA" on the page of the actual problem.** *In the event of an emergency or fire alarm leave your exam on your seat and meet with your GSI or professor outside.*

This exam consists of 4 problems, each of which has parts (a) and (b), in the general topic areas 1) Systems of Linear Equations, 2) Abstract Linear Algebra, 3) Linear Algebra in $\mathbb{R}^n$, 4) Matrix Algebra.

Point values are indicated in brackets to the left of each problem, add up to a total of 80, and so can be used as guide for managing the 80 minute exam time.

Each part of (a) yields full or no credit, and you don't need to show work. *To ensure credit please put each answer (and only the final answer) into the given box.*

Parts (b) can yield partial credit, in particular for explanations and documentation of your approach, even when you don't complete a calculation. In particular, if you recognize your result to be wrong (e.g. by checking!), stating this will yield partial credit. On the other hand, wrong or irrelevant statements mixed with correct work may result in reduced credit.

When asked to explain/show/prove, you should make clear and unambiguous statements that would be accessible to another student. In particular, use words or arrows to indicate how formulas relate to each other. *You may use any theorems or facts stated in the lecture notes, script, and the book sections covered by the course up to Sept.30 – after stating them clearly. If you use theorems or facts that you know from other sources, you will obtain full credit only if you include proofs that derive them from the current course material.*

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1a)

[3] $x_1 - 3x_3 = 8$
$2x_1 - x_2 + 9x_3 = 7$
$x_2 + 5x_3 = -2$
is represented by the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & -1 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

[3] A particular solution for the system represented by the augmented matrix $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -5 & 0 & 6 \\ 0 & 0 & 0 & 1 & -7 \end{array}\right]$

is

$x_1 = -4$

$x_2 = 6$

$\vdots$

$x_3 = 0$

$x_4 = -7$

<u>alternatives</u>

$-3x_3$

$+5x_3$

any other value

$x_1 + 3x_3 = -4$

$x_2 - 5x_3 = 6$

$x_4 = -7$

x_3 free

[4] The reduced echelon form of the matrix $\begin{bmatrix} 0 & 0 & 1 & 2 \\ 4 & 2 & 0 & 0 \\ 2 & 1 & 4 & 8 \end{bmatrix}$ is

$$\begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\sim$

$$\begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 4 & 8 \end{bmatrix}$$

[10] **1b)** Given any 2×3 augmented matrix in reduced echelon form, state conditions which guarantee that the corresponding system of 2 linear equations for 2 variables has infinitely many solutions.

Then make a list of all 2×3 augmented matrices with infinitely many solutions, using the entries 1, 0, or $\star$ (to denote entries that can be any real number).

Existence of solution(s) $\Leftrightarrow$ no pivot in last column

Given existence,

infinitely many solutions $\Leftrightarrow$ free variable(s)

$\Leftrightarrow$ at least one coefficient column without pivot

Together, have 2 coefficient, 1 goal column, at most one pivot-in coefficient column.

$$\left[\begin{array}{cc|c} 1 & * & * \\ 0 & 0 & 0 \end{array}\right] \text{ or } \left[\begin{array}{cc|c} 0 & 1 & * \\ 0 & 0 & 0 \end{array}\right] \text{ or } \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

2a)

[3] The range of a linear transformation $T : V \to W$ is the subspace consisting of

$w \in W$ for which there is a $v \in V$ so that $T(v) = w$

[3] Three vectors $v_1, v_2, v_3 \in V$ are linearly independent if

$c_1v_1 + c_2v_2 + c_3v_3 = 0$ … only for $c_1 = c_2 = c_3 = 0$

[4] If $T : \mathbb{P}_2 \to \mathbb{P}_2$ is a linear tranformation with $T(1 + t^2) = 2t$ and $T(1 - t^2) = 6$, then

$T(t^2) =$ $t - 3$

$$t^2 = \tfrac{1}{2}(1+t^2) - \tfrac{1}{2}(1-t^2)$$

$$\Rightarrow T(t^2) = \tfrac{1}{2}\underbrace{T(1+t^2)}_{2t} - \tfrac{1}{2}\underbrace{T(1-t^2)}_{6}$$

[10] **2b)** Determine the kernel of the linear tranformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ and explain why your computation is true, *using only definitions and the following information (no theorems etc.)*:

- The equation $T(p) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ has solution set $\{p(t) = a_0 + 5t + a_2t^2 \mid a_0, a_2 \in \mathbb{R}\}$.

$\text{Kernel}(T) = \{a_0 + a_2 t^2 \mid a_0, a_2 \in \mathbb{R}\}$

because $a_0 = a_2 = 0$ gives $T(5t) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

and with that

$$
\begin{aligned}
T(q) = 0 &\Leftrightarrow T(q) + T(5t) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
&\Leftrightarrow T(q + 5t) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
&\Leftrightarrow q + 5t \in \text{solution set} \\
&\Leftrightarrow q(t) + 5t = a_0 + 5t + a_2t^2 \text{ for some } a_0, a_2 \in \mathbb{R} \\
&\Leftrightarrow q(t) = a_0 + a_2t^2 \quad \text{— " —}
\end{aligned}
$$

This proves both directions at once. Alternatively, you could make separate arguments for

- $q(t) = a_0 + a_2t^2 \Rightarrow T(q) = 0$
- $T(q) = 0 \Rightarrow q(t) = a_0 + a_2t^2$ for some $a_0, a_2 \in \mathbb{R}$

[3] **3a)** The linear transformation $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+2y \\ 4y \end{bmatrix}$ is represented by the matrix $\boxed{\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}}$.

To compute standard matrix:

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

[4] The vectors $\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$ are ... [check all that apply] ...

- ☐ spanning $\mathbb{R}^3$
- ☐ a basis of $\mathbb{R}^3$
- ☒ linearly dependent
- ☐ linearly independent

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & -3 \\ 4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{row of } 0$$

↑ free variable

[3] Circle each subspace of $\mathbb{R}^2$; cross out sets that are not subspaces of $\mathbb{R}^2$.

$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; y = 3x \right\}$ = Kernel of $\mathbb{R}^2 \to \mathbb{R}$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto y - 3x$

OR closed under scaling and addition, and contains $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\cancel{\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; y = 3x^2 \right\}}$ contains $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ but not $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ because $6 \neq 3 \cdot 2^2$

$\cancel{\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; y = 3x + 1 \right\}}$ does not contain $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

[10] **3b)** Find the set of solutions $\mathbf{x} \in \mathbb{R}^3$ of the equation $\begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 10 \end{bmatrix}$.

Then state a general solution principle for inhomogeneous linear equations (no need to prove it) and explain how your result is an example of this principle.

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ -2 & 4 & 1 & -1 \\ 0 & 0 & 2 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \Updownarrow$$

$$x_1 - 2x_2 = 3$$
$$x_3 = 5$$
$$x_2 \text{ free}$$

The solution set

$$\left\{ \underline{x} = \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; x_2 \in \mathbb{R} \right\}$$

is of the form $\underline{p}$ + kernel with $\underline{p} = \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$

and $\left\{ x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; x_2 \in \mathbb{R} \right\}$ is the kernel of $\underline{x} \mapsto \begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} \underline{x}$

<u>General Solution Principle:</u> Given $T: V \to W$ linear and $b \in W$

$$\{\text{solutions of } T(x) = b\} = p + \{\text{solutions of } T(x) = 0\}$$

for p any "particular" solution of $T(p) = b$.

[5] **4a)** Compute or state "not defined" for the products of $A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 2 & 8 \\ 5 & 2 & 3 \\ +6 & +3 & +12 \end{bmatrix}$$

$AB = \boxed{\begin{bmatrix} 4 & 2 & 8 \\ 11 & 5 & 15 \end{bmatrix}}$

$BA = \boxed{\text{not defined}}$

[5] $\det \begin{bmatrix} 2 & 0 & 0 & 3 \\ 3 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 3 & 2 \end{bmatrix} = \boxed{-6}$

$$= -1 \det \begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix} = -1 \cdot 2 \cdot 1 \cdot 3$$

[10] **4b)** Compute the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & -2 \end{bmatrix}$.

Then state a definition of "inverse matrix" and explain why it provides a general formula for solutions $\mathbf{x}$ of the equation $A\mathbf{x} = \mathbf{b}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3/2 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 3/2 & -1/2 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 3/2 & -1/2 \end{bmatrix}$$

By definition, $A\underline{x} = \underline{b} \Leftrightarrow \underline{x} = A^{-1}\underline{b}$, so the solution set to $A\underline{x} = \underline{b}$ is $\{\underline{x} = A^{-1}\underline{b}\}$.

Alternative definition: $AA^{-1} = I$ gives $\underline{x} = A^{-1}\underline{b} \Rightarrow A\underline{x} = \underline{b}$

$A^{-1}A = I$ gives $A\underline{x} = \underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b}$

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