1A	
1B	
2A	
2B	
3	
Total	

1A Let x_n be a sequence of real numbers defined by $x_0 = 2$ and

$$x_{n+1} = 1 + \frac{1}{x_n} = g(x_n).$$

Assume $x_n \to x$ for some $x \geq \sqrt{2}$ as $n \to \infty$. Don't find x. Show that $\sqrt{2} \leq x_n \leq 2$ and

$$|x_{n+1} - x| \leq \frac{1}{2}|x_n - x|$$

for all n.

Solution:

If $x_n \leq 2$ then $1/x_n \geq 1/2$ so $x_{n+1} \geq 3/2 \geq \sqrt{2}$.
If $x_n \geq \sqrt{2}$ then $1/x_n \leq 1/\sqrt{2}$ so $x_{n+1} \leq 1+1/\sqrt{2} \leq 2$ (since $1/\sqrt{2} \leq 0.8$).
Since $x_0 = 2$ we have $\sqrt{2} \leq x_n \leq 2$ for all n.
By continuity, $x = 1 + 1/x$ so

$$x_{n+1} - x = \frac{1}{x_n} - \frac{1}{x} = \frac{x - x_n}{x x_n}.$$

Since $x \geq \sqrt{2}$ and $x_n \geq \sqrt{2}$ we have

$$|x_{n+1} - x| \leq \frac{1}{2}|x_n - x|$$

for all n.

Rubric:

Invariance: 10 pts.
Contractivity: 10 pts.

1B In floating point arithmetic, x_n is approximated by y_n satisfying

$$y_{n+1} = \mathrm{fl}(x_{n+1}) = \left(1 + \frac{1}{y_n}(1+\delta_n)\right)(1+\delta_n')$$

where $|\delta_n| \leq \epsilon$ and $|\delta_n'| \leq \epsilon$. Assume that $\sqrt{2} \leq y_n \leq 2$. Show that

$$|y_{n+1} - x| \leq \frac{1}{2}|y_n - x| + 3\epsilon + O(\epsilon^2)$$

for all n, and describe the behavior of y_n as $n \to \infty$.

Solution:

Simplifying gives

$$y_{n+1} = 1 + \frac{1}{y_n} + \frac{1}{y_n}\delta_n + \left(1 + \frac{1}{y_n}\right)\delta_n' + O(\epsilon^2),$$

and subtracting from $x = 1 + 1/x$ gives

$$y_{n+1} - x = \frac{x - y_n}{x y_n} + \delta_n' + \frac{2}{y_n}\delta_n + O(\epsilon^2).$$

Since $x \geq \sqrt{2}$ and $y_n \geq \sqrt{2}$,

$$|y_{n+1} - x| \leq \frac{1}{2}|x - y_n| + (1 + \sqrt{2})\epsilon + O(\epsilon^2) \leq \frac{1}{2}|x - y_n| + 3\epsilon + O(\epsilon^2).$$

As $n \to \infty$, the error $y_n - x$ will decrease until it reaches $O(\epsilon)$. When $|y_n - x| = a\epsilon$ reaches its minimum, then a will satisfy

$$a\epsilon = \frac{1}{2}a\epsilon + 3\epsilon + O(\epsilon^2)$$

so $a = 6$. Thus the minimum possible error will be about 6ϵ, and will be reached in about 50 steps.

Rubric:

Error bound: 10 pts.
Analysis: 10 pts.

2A For an arbitrary function f, let $H(t)$ be the quadratic polynomial interpolating $f(1)$, $f(2)$, and $f'(2)$. Give a formula for the error $f(t) - H(t)$ and explain why each of the three factors in your formula is inevitable.

Solution:

The error is

$$f(t) - H(t) = \frac{f'''(\xi)}{3!}(t-1)(t-2)^2$$

where

the third derivative ensures that the error vanishes when f is a quadratic polynomial (and thus exactly reproduced by the uniqueness of polynomial interpolation),

the polynomial $(t-1)(t-2)^2$ ensures that the error vanishes at $t = 1$ and $t = 2$, and its derivative also vanishes at $t = 2$,

and the 3! ensures that the error formula is correct when we try it on $f(t) = (t-1)(t-2)^2 = t^3 + \cdots$, where $H(t) = 0$ and $f''' = 3!$.

Rubric:

Error formula: 5 pts.

Explanation: 5 pts per factor.

2B For $f(t) = 1/t$, build the divided difference table, find the Newton form of $H(t)$, and show that the error $|f(t) - H(t)| \le 4/27$ on the interval $0 \le t \le 1$.

Solution:

The difference table is constructed with $f'(t_j)/1! = -1/t_j^2$ in place of $f[t_j, t_{j+1}]$ whenever $t_{j+1} = t_j$:

j	t_j	$f[t_j]$	$f[t_j, t_{j+1}]$	$f[t_j, t_{j+1}, t_{j+2}]$
0	1	1	$\frac{-1}{2}$	$\frac{1}{4}$
1	2	$\frac{1}{2}$	$\frac{-1}{4}$	
2	2	$\frac{1}{2}$		

Thus reading along the top row gives the Newton form of $H(t)$:

$$H(t) = 1 - \frac{1}{2}(t-1) + \frac{1}{4}(t-1)(t-2).$$

(Check that $H(1) = 1$, $H(2) = 1/2$ and $H'(2) = -1/4$.)

Since $|f'''(t)| = |-6/t^3| \le 6$ on $1 \le t \le 2$,

$$|f(t) - H(t)| \le \frac{6}{3!}|(t-1)(t-2)^2|.$$

The polynomial $(t-1)(t-2)^2$ has an extremum at $t = 2$ and another at the point where $(t-2)^2 + 2(t-1)(t-2) = 0$ or $t = 4/3$. At $t = 4/3$ the value is $4/27$ so the error is bounded by $4/27$ on the interval $1 \le t \le 2$.

Rubric:

Difference table: 5 pts.
Newton form: 5 pts.
Error bound: 10 pts.

3 Find constants a, b and c such that the numerical integration rule

$$\int_{-1}^{1} f(t)\ dt = af(-1) + bf(0) + cf(1)$$

is exact whenever f is a quadratic polynomial. Show that the error is bounded by $M_3/12$ whenever the third derivative $|f'''(t)| \leq M_3$ for $|t| \leq 1$.

Solution:

By Lagrange interpolation,

$$a = \int_{-1}^{1} L_{-1}(t)dt = \int_{-1}^{1} \frac{(t-0)(t-1)}{(-1-0)(-1-1)}dt = \frac{1}{3}$$

and

$$b = \int_{-1}^{1} L_{0}(t)dt = \int_{-1}^{1} \frac{(t+1)(t-1)}{(0+1)(0-1)}dt = \frac{4}{3}.$$

Since the weights must sum to 2, we have also

$$c = \frac{1}{3}.$$

Using the error formula for polynomial interpolation, the error is bounded by

$$|\int_{-1}^{1} \frac{f'''(\xi)}{3!}(t+1)(t-0)(t-1)dt| \leq \frac{M_3}{6}\int_{-1}^{1} |(t+1)(t-0)(t-1)|dt.$$

Note that the integrand changes sign on the interval of integration. Hence the error is bounded by

$$\frac{M_3}{6} 2\int_{0}^{1} t - t^3 dt = \frac{M_3}{12}.$$

Rubric:

Weights: 10 pts.
Error bound: 10 pts.