NAME: **STUDENT ID:**

Math 54 , second midterm, Fall 2018, John Lott

This is a closed book, closed notes, closed calculator, closed computer, closed phone, closed network, open mind exam.

Be sure to write your name and student id number on the top of EVERY page that you turn in.

Name of neighbor to your left ______________________________

Name of neighbor to your right ______________________________

Write your answers in the boxes provided. For full credit, show your work, too, (except for the true/false questions) and cross out work that you do not want us to grade.

You can write on the backs of pages. You can also use the back of this page, and the back of the last page, as scratch paper. If you need additional scratch paper, ask me for it.

NAME: **STUDENT ID:**

1. (11 pts) Apply the Gram-Schmidt procedure to the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

to generate an orthogonal set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Write the vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ in the box.

$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$, which we might as well replace by

$\mathbf{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} \mathbf{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

NAME: **STUDENT ID:**

2. (15 pts) (No partial credit. To get credit, you must show your work.)

a. Find the eigenvalues of

$$\begin{bmatrix} 3 & 4 & 5 \\ 0 & 2 & 7 \\ 0 & 7 & 2 \end{bmatrix}.$$

Write them in ascending order in the box.

$$\boxed{}$$

$$\mathrm{DET}(A - \lambda I) = \mathrm{DET}\begin{bmatrix} 3-\lambda & 4 & 5 \\ 0 & 2-\lambda & 7 \\ 0 & 7 & 2-\lambda \end{bmatrix} = (3-\lambda)\left((2-\lambda)^2 - 49\right).$$

One root is $\lambda = 3$. The other roots satisfy $(\lambda - 2)^2 = 49$, i.e. $\lambda - 2 = \pm 7$. The answer is $\{-5, 3, 9\}$.

NAME: **STUDENT ID:**

b. Find the rank of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 1 & 4 & 7 & 10 \end{bmatrix}.$$

Write it in the box.

$$\boxed{}$$

Row reduction gives

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are two pivot columns, so the answer is 2.

NAME: **STUDENT ID:**

c. Find the closest point to $\begin{bmatrix} -1 \\ 5 \\ 10 \end{bmatrix}$, in the span of $\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. Write it in the box.

$$\frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{-5}{30} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \frac{-1}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

NAME: **STUDENT ID:**

3. (Note : you can do part b of this problem without doing part a.)

If A is a square matrix then the *trace* of A, denoted $\mathrm{tr}(A)$, is the sum of its diagonal entries.

Fact : If A and B are $n \times n$ matrices then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

a. (8 pts) Using the fact above, show that if S and T are similar matrices then $\mathrm{tr}(S) = \mathrm{tr}(T)$. Carefully explain your reasoning.

b. (8 pts) Using part a, show that if A is a diagonalizable square matrix then $\mathrm{tr}(A)$ equals the sum of the eigenvalues of A.

a. Since S and T are similar, there is some invertible matrix P so that $S = PTP^{-1}$. Then $\mathrm{tr}(S) = \mathrm{tr}(PTP^{-1}) = \mathrm{tr}(P(TP^{-1})) = \mathrm{tr}((TP^{-1})P) = \mathrm{tr}(T(P^{-1}P)) = \mathrm{tr}(TI) = \mathrm{tr}(T)$.

b. If A is diagonalizable then A is similar to the diagonal matrix D consisting of the eigenvalues $\{\lambda_i\}_{i=1}^n$ of A. From part a, $\mathrm{tr}(A) = \mathrm{tr}(D) = \sum_{i=1}^n \lambda_i$.

NAME: **STUDENT ID:**

4. (18 pts) Check only one box per question. No explanation is necessary.
WARNING: there are multiple versions of this problem. If you copy off of a neighbor, you will probably get around half of the answers wrong.

If A is diagonalizable then so is A^2.

☐ TRUE ☐ FALSE

If the square matrix A fails to be invertible then 0 must be an eigenvalue of A.

☐ TRUE ☐ FALSE

If A is a square matrix and the characteristic polynomial of A is $(\lambda - 6)^2(\lambda - 7)^2$ then there exist two linearly independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $A\mathbf{v}_1 = 6\mathbf{v}_1$ and $A\mathbf{v}_2 = 6\mathbf{v}_2$.

☐ TRUE ☐ FALSE

If S is a linearly independent set of vectors in a vector space V then S is a basis for Span(S).

☐ TRUE ☐ FALSE

If A and B are matrices so that $AB = I_n$ then A and B are both invertible.

☐ TRUE ☐ FALSE

NAME: **STUDENT ID:**

Each eigenvector of an invertible matrix A is also an eigenvector of A^{-1}.

☐ <u>TRUE</u> ☐ FALSE

The matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is diagonalizable.

☐ TRUE ☐ <u>FALSE</u>

If A is an orthogonal $n \times n$ matrix then $Row(A) = Col(A)$.

☐ <u>TRUE</u> ☐ FALSE

The least squares solution of $A\mathbf{x} = \mathbf{b}$ is the point in the column space of A which is closest to $\mathbf{b}$.

☐ TRUE ☐ <u>FALSE</u>