Problem 1  (50 points)

(a) By pattern-matching the coefficients,

\[(i\omega RC + 1)H(\omega) = i\omega RC - 1 \implies RC\dot{y} + y = RC\dot{x} - x\]

Giving us the desired linear, constant-coefficient differential equation.

\[
\dot{y} = -\frac{1}{RC}y + \dot{x} - \frac{1}{RC}x
\]

In creating the integrator-adder-gain block diagram implementation of the filter, it will be illustrative to integrate the differential equation:

\[
y = -\frac{1}{RC}\int y + \dot{x} - \frac{1}{RC}\int x
\]

(b) \[H(\omega) = \frac{i\omega RC - 1}{i\omega RC + 1} = \frac{i\omega - \frac{1}{RC}}{i\omega + \frac{1}{RC}} = -\frac{1}{RC} - i\omega
\]

Observing that the numerator is the complex conjugate of the denominator, we find that \[|H(\omega)| = 1\]

This is an all-pass filter. Solving for the phase response, we have

\[
\angle H(\omega) = \angle(-1) + \angle\left(\frac{1}{RC} - i\omega\right) - \angle\left(\frac{1}{RC} + i\omega\right)
\]

\[= \pi - 2\tan^{-1}(RC\omega)\]
The plot is as follows, where the $x$-axis is $\omega$, the $y$-axis is $\angle H(\omega)$, the $y$-intercept is $\pi$, the left asymptote is $2\pi$, and the right asymptote is $0$.

![Plot](image)

(c) 

\[ G(\omega) = \int_{-\infty}^{\infty} \frac{df(t)}{dt} e^{-i\omega t} \, dt = \int_{-\infty}^{\infty} e^{-i\omega t} \, df(t) \]

Using integration by parts, where $u = e^{-i\omega t}$ and $dv = df(t)$, such that $du = -i\omega e^{-i\omega t} \, dt$ and $v = f(t)$, we have

\[ G(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \, df(t) = f(t) e^{-i\omega t} \bigg|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt = 0 + i\omega F(\omega) \]

Where the first term goes to $0$ because we assume $\lim_{t \to \pm\infty} f(t) = 0$, a clarification made during the examination. The second term can be recognized from the analysis equation, with a coefficient of $i\omega$ falling out of the $du$ term in integration by parts.

(d) 

\[ H(\omega) = \frac{i\omega - \frac{1}{RC}}{i\omega + \frac{1}{RC}} = i\omega \left( \frac{1}{i\omega + \frac{1}{RC}} \right) - \frac{1}{RC} \frac{1}{i\omega + \frac{1}{RC}} \]

Observing the Fourier pair

\[ e^{-\frac{t}{RC}} u(t) \xleftrightarrow{F} \frac{1}{i\omega + \frac{1}{RC}} \]

We know that

\[ -\frac{1}{RC} e^{-\frac{t}{RC}} u(t) \xleftrightarrow{F} -\frac{1}{RC} \frac{1}{i\omega + \frac{1}{RC}} \]

Using part (c), we find that

\[ i\omega \left( \frac{1}{i\omega + \frac{1}{RC}} \right) \xleftrightarrow{d} \frac{d}{dt} \left( e^{-\frac{t}{RC}} u(t) \right) = \delta(t) - \frac{1}{RC} e^{-\frac{t}{RC}} u(t) \]
Hence $H(\omega)$ must be the frequency response of $h(t) = \delta(t) - \frac{2}{RC} e^{-\frac{t}{RC}} u(t)$. 

\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (2,0);
\draw[->] (0,0) -- (0,-2);
\node at (1,0) {$0$};
\node at (0,-1) {$0$};
\node at (0,-2) {$-\frac{2}{RC}$};
\node at (1,1) {$h(t)$};
\end{tikzpicture}
\end{center}
Problem 2 (60 points)

(a) 
\[ y(n) = \alpha y(n-1) + x(n) - x(n-1) \]

(b) We present two approaches.

Method 1:
\[ f(n) = \alpha f(n-1) + \delta(n) - \delta(n-1) \]
\[ f(0) = \alpha f(-1) + \delta(0) - \delta(-1) = 1 \]
\[ f(1) = \alpha f(0) + \delta(1) - \delta(0) = \alpha - 1 \]
\[ f(2) = \alpha f(1) = \alpha(\alpha - 1) \]
\[ f(3) = \alpha f(2) = \alpha^2(\alpha - 1) \]
\[ \vdots \]
\[ f(n) = \alpha^{n-1}(\alpha - 1) \]

\[ f(n) = \alpha^n u(n) - \alpha^{n-1} u(n-1) = \begin{cases} 
0 & n < 0 \\
1 & n = 0 \\
\alpha^{n-1}(\alpha - 1) & n \geq 1 
\end{cases} \]

Method 2:
\[ F(\omega) = \frac{1 - e^{-i\omega}}{1 - \alpha e^{-i\omega}} = \frac{1}{1 - \alpha e^{-i\omega}} - \frac{e^{-i\omega}}{1 - \alpha e^{-i\omega}} \]

Using the time-shift property of the Fourier transform and the Fourier pair
\[ \frac{1}{1 - \alpha e^{-i\omega}} \leftrightarrow \alpha^n u(n) \]

We have
\[ f(n) = \alpha^n u(n) - \alpha^{n-1} u(n-1) \]

(c) Passing in input \( x(n) = e^{i\omega n} \) into the given LCCDE, we obtain
\[ F(\omega)e^{i\omega n} - \alpha e^{-i\omega} F(\omega)e^{i\omega n} = e^{i\omega n} - e^{-i\omega} e^{i\omega n} \]
Dividing by the complex exponential and grouping terms, we have
\[(1 - \alpha e^{-i\omega}) F(\omega) = 1 - e^{-i\omega} \implies F(\omega) = \frac{1 - e^{-i\omega}}{1 - \alpha e^{-i\omega}}.\]

(d)
\[F(\omega) = \frac{e^{i\omega} - 1}{e^{i\omega} - 0.99}\]

If \(\alpha\) is close to 1, as is the case here, the two vectors representing \(e^{i\omega} - 1\) and \(e^{i\omega} - \alpha\) are nearly identical for \(e^{i\omega}\) not in the close neighborhood of 1, so we find that \(|F(\omega)| \approx 1\). This is a notch filter with notch at \(\omega = 0\).

(e)

(i)
\[x(n) = \frac{1}{2}e^{i\omega n} + \frac{1}{2}e^{i\pi n} \implies y(n) = \frac{F(0)}{2}e^{i\omega n} + \frac{F(\pi)}{2}e^{i\pi n} = \frac{F(\pi)}{2}e^{i\pi n}\]

From the expression in the previous part, we know \(F(\pi) = \frac{2}{1 + \alpha}\), so
\[y(n) = \frac{(-1)^n}{1 + \alpha}\]

(ii)
\[\alpha = 0.99 \implies F(\pi) \approx 1 \implies y(n) \approx \frac{(-1)^n}{2}\]

The plot is an alternating sequence of \(\pm \frac{1}{2}\), with \(y(0) = \frac{1}{2}\).

**Problem 3** We know that when \(y(n) = \delta(n), r(n) = g(n)\). If we can find an input \(\hat{x}(n)\) such that the corresponding \(\hat{y}(n) = \delta(n)\), then this \(\hat{x}(n) = g(n)\), since \(x(n) = r(n)\). From plotting the given input-output pair, we can observe that \(y(n) - \alpha y(n - N) = \delta(n)\). Since \(\hat{F}\) is an LTI system, \(\hat{x}(n) = x(n) - \alpha x(n - N)\) is the impulse response \(g(n)\) of the system \(\hat{F}\). That is,
\[g(n) = \delta(n) + \delta(n - 1) - \alpha \delta(n - N) - \alpha \delta(n - N - 1)\]

And so we can read off the frequency response as
\[G(\omega) = 1 + e^{-i\omega} - \alpha e^{-i\omega N} - \alpha e^{-i\omega(N+1)}\]

**Problem 4** (50 points)
(a) 

\[ y(t) = \int_{-\infty}^{\infty} h(\sigma) x(t - \sigma) d\sigma = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t - \sigma) d\sigma \]

By a change of variables \( \tau = t - \sigma \), such that \( \sigma = t - \tau \) and \( d\sigma = -d\tau \),

\[ = \frac{1}{T} \int_{t+\frac{T}{2}}^{t-\frac{T}{2}} x(\tau)(-d\tau) = \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} x(\tau) d\tau \]

Then we can read off the values

\[ A = \frac{T}{2} \]

\[ B = T \]

This is a moving average filter over a time window \( T \).

(b) Observing that \( \lim_{T \to 0} h(t) = \delta(t) \), the system approaches the identity system.

\[ \lim_{T \to 0} y(t) = x(t) \]

(c)

\[ H(\omega) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i\omega t} dt \]

\[ = \frac{1}{T} \left| e^{-i\omega \frac{T}{2}} - e^{-i\omega \frac{T}{2}} \right| \]

\[ = \frac{\omega T}{i \omega T} \left| e^{\frac{i\omega T}{2}} - e^{-\frac{i\omega T}{2}} \right| \]

\[ = \frac{\omega T}{i \omega T} \frac{2}{2i} \]

\[ = \frac{\sin (\omega \frac{T}{2})}{\frac{\omega T}{2}} \]

From which we see that \( C = \frac{T}{2} \) and \( D = \frac{T}{2} \).
(i) \[ y(t) = \begin{cases} \frac{T + t}{T^2} & -T \leq t \leq 0 \\ \frac{T - t}{T^2} & 0 < t \leq T \\ 0 & |t| > T \end{cases} \]

(ii) \[ \omega_0 = \frac{4\pi}{T} \implies T_0 = \frac{2\pi}{\omega_0} = \frac{T}{2} \]

Since the cosine undergoes two cycles over a duration of \( T \),
\[ y(t) = 0 \]

(iii)

(iv) \[ y(t) = \frac{1}{T} \int_{t-T/2}^{t+T/2} x(\tau) d\tau \]

Case 1: \( t < -\frac{T}{2} \implies y(t) = 0 \)

Case 2: \( -\frac{T}{2} \leq t \leq \frac{T}{2} \)
\[ y(t) = \frac{1}{T} \int_{0}^{t+T/2} e^{-\tau} d\tau = -\frac{e^{-\tau}}{T} \bigg|_{0}^{t+T/2} = \frac{1 - e^{-(t+T/2)}}{T} \]

Case 3: \( t > \frac{T}{2} \)
\[ y(t) = \frac{1}{T} \int_{t-T/2}^{t+T/2} e^{-\tau} d\tau = -\frac{1}{T} e^{-\tau} \bigg|_{t-T/2}^{t+T/2} = \frac{e^{-(t-\frac{T}{2})} - e^{-(t+\frac{T}{2})}}{T} = e^{-T} \left( e^{\frac{T}{2}} - e^{-\frac{T}{2}} \right) \]
\[ y(t) = \frac{1 - e^{-T}}{T} \]