# Midterm I

1. Define the following terms. Be as precise as you can.

   (a) An open subset of a metric space.
   A subset $U$ of a metric space is open if every for $p \in U$, there exists an open ball around $p$ contained in $U$.

   (b) A compact subset of a metric space.
   A compact subset of a metric space is a set for which every open cover has a finite subcover.

   (c) A convergent sequence in a metric space.
   A sequence $x.$ coverges to $x$ if $x.$ is eventually in every neighborhood $W$ of $x$, that is, if for every $W$, there exists an $N$ such that $x_n \in W$ for all $n \geq N$.

   (d) A limit point of a subset of a metric space.
   A point $x$ is a limit point of $E$ if every neighborhood of $x$ meets $E$ in some point other than $x$.

2. Give an example of the following, or prove that no such example exists. If you give an example, prove that it works.

   (a) A nonempty bounded subset of the reals with no maximum.
   The set $(0,1)$ is nonempty and bounded and has no maximum. Indeed, if $x \in (0,1)$, then $x < 1$, but $\frac{1+x}{2} \in (0,1)$ and is greater than $x$.

   (b) An uncountable subset of the reals with no limit point.
   Every uncountable subset $E$ of $\mathbf{R}$ has a limit point. Indeed, if $E$ is such a set, then for some $n$, the set $E_n := E \cap [n, n+1]$ is infinite. Since $[n, n+1]$ is compact, $E_n$ has a limit point.

   (c) A nonempty proper subset of the closed interval $[0,1]$ which is both open and closed. Do not use a theorem which makes the proof trivial.
   No such set exists. Suppose $A$ is a nonempty subset of $[0,1]$ which is both open and closed. Since it is nonempty and bounded, it has a supremum, call it $c$. Since $A$ is closed, $c \in A$. If $c < 1$, then since $A$ is open, it contains a neighborhood of $c$, hence a point which is larger than $c$, a contradiction. This shows that $1 \in A$. If the complement of $A$ were not empty, the same argument would show that it would also contain 1. Contradiction.

3. (Zeno). Suppose the sequence $x.$ is defined inductively as follows. $x_0 = 0$ and $x_{n+1}$ is half way between $x_n$ and 1.

(a) Prove that the limit of $x.$ exists.
It is clear that $x.$ is an increasing and bounded sequence, hence has a limit.

(b) Evaluate the limit of $x.$.
This limit must be 1.

(c) Prove that your answer in part b is correct.
We have $x_{n+1} = \frac{x_n+1}{2}$. Since limits are compatible with sums and subsequences, $x = \frac{x+1}{2}$, where $x$ is the limit. Hence $x = 1$.