1. (a) **(4 pts)** Letting $a_k := \sup\{x_n : n \geq k\}$ for each $k \in \mathbb{N}$ we have

$$\limsup_{n\to\infty} x_n = \lim_{k\to\infty} a_k$$

Note that since a_k is a decreasing sequence, and is bounded below (by any lower bound for $(x_n)_{n\in\mathbb{N}}$). Hence it converges and in fact its limit is $\inf\{a_k : k \in \mathbb{N}\}$.

(b) **(6 pts)** We first note that $(x_n)_{n\in\mathbb{N}}$ being bounded implies a_k as above exists for each $k \in \mathbb{N}$. Since S is closed, $a_k \in S$. Indeed, otherwise a_k is contained in the complement of S_k, which is open. This means there exists $r > 0$ such that $(a_k - r, a_k + r) \subset S^c$. However, as the supremum of $\{x_n : n \geq k\}$, we know there exists $n \geq k$ such that

$$a_k - r < x_n \leq a_k,$$

which contradicts $x_n \in S$. Thus $a_k \in S$. Now, by definition

$$\limsup_{n\to\infty} x_k = \lim_{k\to\infty} a_k.$$

Since $(a_k)_{k\in\mathbb{N}} \subset S$ and S is closed, it necessarily contains this limit. □

2. (a) **(3 pts)** The sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy if $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $d(x_n, x_m) < \epsilon$.

(b) **(2 pts)** A metric space is complete if every Cauchy sequence converges.

(c) **(5 pts)** Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (E, d). We claim that $(x_n)_{n\in\mathbb{N}}$ is eventually constant and hence necessarily converges. Indeed, for $\epsilon = \frac{1}{2}$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$d(x_n, x_m) < \frac{1}{2}.$$

However, since the metric only takes values of 0 and 1, it must be that $d(x_n, x_m) = 0$ for all $n, m \geq N$. That is, $x_n = x_m = x_N$ for all $n, m \geq N$. Thus the sequence clearly converges to x_N. Since this Cauchy sequence was arbitrary, we see that (E, d) is complete. □

3. (a) **(4 pts)** A subset S is compact if every open cover has a finite subcover; that is, whenver $\{U_i\}_{i\in I}$ is a collection of open subsets of E satisfying

$$S \subset \bigcup_{i\in I} U_i,$$

then there are $i_1, \ldots, i_n \in I$, for some $n \in \mathbb{N}$, such that

$$S \subset U_{i_1} \cup \cdots \cup U_{i_n}.$$

Alternatively, S is sequentially compact: every sequence $(x_n)_{n\in\mathbb{N}} \subset S$ has a convergent subsequence in S.

Alternatively, S is complete and totally bounded: S is complete and given any $\epsilon > 0$ we can cover S by finitely many closed balls of radius ϵ.

(b) **(6 pts)** If S is finite, write $S = \{s_1, \ldots, s_n\}$. Let $\{U_i\}_{i\in I}$ be an open cover for S. For each $j = 1, \ldots, n$ we can find $i_j \in I$ so that $s_j \in U_{i_j}$. Then $\{U_{i_1}, \ldots, U_{i_n}\}$ is an open cover for S and hence S is compact. If S is infinite, consider the open cover given by

$$\left\{B(s, \frac{1}{2})\right\}_{s\in S}.$$

This is an infinite collection and there are no finite subcovers. In fact, there are not even any subcovers: $s' \in B(s, \frac{1}{2})$ if and only if $s' = s$ and hence the collection cannot cover S is any of the open balls are removed. □

Alternatively, since S is infinite we can define a sequence $(x_n)_{n\in\mathbb{N}} \subset S$ so that $x_n \neq x_m$ for any distinct indices $n, m \in \mathbb{N}$. Consequently, $d(x_n, x_m) = 1$ for all $n \neq m$. Then no subsequence can converge because it will not even be a Cauchy sequence. □

4. (a) **(4 pts)** The function f is continuous if for each $x \in E$ and for any $\epsilon > 0$ there exists $\delta > 0$ so that if $y \in E$ satisfies $d(x, y) < \delta$ then $d'(f(x), f(y)) < \epsilon$.

Alternatively, $f^{-1}(U) \subset E$ is open for every open set $U \subset E'$.

Alternatively, whenever $(x_n)_{n\in\mathbb{N}} \subset E$ converges to $x \in E$, we have that $(f(x_n))_{n\in\mathbb{N}} \subset E'$ converges to $f(x) \in E'$.

(b) **(6 pts)** We note that

$$f(x,y) = d_2((x,y),(0,0))^2.$$

On the homework we show that $(x,y) \mapsto d_2((x,y),(0,0))$ is continuous. In class we showed $t \mapsto t^2$ is continuous. Since the composition of continuous functions is continuous, we have that f is continuous. □

Alternatively, fix $(x_0,y_0) \in \mathbb{R}^2$ and let $\epsilon > 0$. Let

$$\delta = \min\left\{1, \frac{\epsilon}{2(2|x_0|+1)}, \frac{\epsilon}{2(2|y_0|+1)}\right\}.$$

Suppose $(x,y) \in \mathbb{R}^2$ satisfies $d_2((x,y),(x_0,y_0)) < \delta$. Note that this implies $|x-x_0|, |y-y_0| < \delta$. Moreover, it follows that $|x| \leq |x_0| + 1$ and $|y| \leq |y_0| + 1$. We estimate:

$$\begin{aligned} d(f(x,y), f(x_0,y_0)) &= |x^2 + y^2 - x_0^2 - y_0^2| \\ &\leq |x^2 - x_0^2| + |y^2 - y_0^2| \\ &\leq |x - x_0|(|x| + |x_0|) + |y - y_0|(|y| + |y_0|) \\ &< \delta(2|x_0| + 1) + \delta(2|y_0| + 1) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus f is continuous at (x_0,y_0). Since $(x_0,y_0) \in \mathbb{R}^2$ was arbitrary, we see that f is continuous on all of $\mathbb{R}^2$. □