UCB Math 121B, Spring 2018: Midterm Exam 1

Prof. Persson, February 20, 2018

Name: Solutions

SID: ____________________

Grading	
1	/ 5
2	/ 4
3	/ 4
4	/ 4
5	/ 4
	/21

Instructions:

- One sheet of notes, no books, no calculators.
- Exam time 80 minutes, do all of the problems.
- You must justify your answers for full credit.
- Write your answers in the space below each problem.
- If you need more space, use reverse side or scratch pages. Indicate clearly where to find your answers.

1. Evaluate the integrals explicitly (write answer in terms of π):

a) (3 points) $\int_0^\infty \sqrt{x} e^{-x^{1/3}}\, dx$ $=$ $\left[\begin{array}{c|c|c} u = x^{1/3} & dx = 3u^2 du & x=0 \Leftrightarrow u=0 \\ x = u^3 & \sqrt{x} = u^{3/2} & x=\infty \Leftrightarrow u=\infty \end{array} \right]$

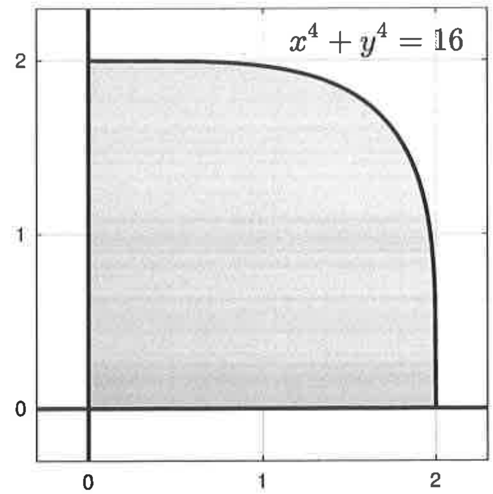
$= \int_0^\infty u^{3/2} e^{-u}\, 3u^2 du = 3\int_0^\infty u^{7/2} e^{-u} du = 3\Gamma\left(\frac{9}{2}\right)$

$= 3 \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \underline{\underline{\frac{315}{16}\sqrt{\pi}}}$

b) (2 points) $\int_0^{\pi/2} \sin^6 \theta\, d\theta$ $= \frac{1}{2} B\left(\frac{7}{2}, \frac{1}{2}\right) = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)}$

$= \frac{1}{2} \cdot \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \pi}{3!} = \underline{\underline{\frac{5\pi}{32}}}$

2. (4 points) Consider the graph of $x^4 + y^4 = 16$. Write the integral for the first quadrant area bounded by the curve, and evaluate it in terms of π and the Γ function.



$$A = \int_0^2 (16 - x^4)^{1/4}\,dx$$

$$= 2\int_0^2 \left(1 - \left(\frac{x}{2}\right)^4\right)^{1/4} dx$$

$$= \left[\begin{array}{l|l|l} \dfrac{x^4}{16} = u & 4x^3\,dx = 16\,du & x = 0 \iff u = 0 \\ x = 2u^{1/4} & dx = \dfrac{4}{x^3}\,du = \dfrac{1}{2}u^{-3/4}\,du & x = 2 \iff u = 1 \end{array}\right]$$

$$= 2\int_0^1 (1-u)^{1/4} \cdot \frac{1}{2} u^{-3/4}\,du = B\left(\frac{1}{4}, \frac{5}{4}\right)$$

$$= \frac{\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\cdot\frac{1}{2}\cdot\sqrt{\pi}} = \underline{\underline{\frac{\Gamma\left(\frac{1}{4}\right)^2}{2\sqrt{\pi}}}}$$

3. (4 points) Find the general solution of the differential equation

$$xy' - (1 + 3x)y = 0$$

using power series.

$$\begin{cases} y = \sum_{n=0}^{\infty} a_n x^n \\ y' = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1} \end{cases}$$

$$\Rightarrow \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n - 3\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

x^0: $\quad -a_0 = 0 \quad \Rightarrow \quad a_0 = 0$

x^1: $\quad a_1 - a_1 - 3a_0 = 0$

$\quad 0 = 0 \quad \Rightarrow \quad a_1 = A$

x^n: $\quad n a_n - a_n - 3a_{n-1} = 0$

$$a_n = \frac{3}{n-1} a_{n-1} = \frac{3^{n-1}}{(n-1)!} A$$

$$\Rightarrow y = A \sum_{n=1}^{\infty} \frac{3^{n-1}}{(n-1)!} x^n = \underline{\underline{A \sum_{n=0}^{\infty} \frac{3^n}{n!} x^{n+1}}} \quad \left(= A x e^{3x} \right)$$

4. (4 points) Expand the following function in Legendre series (up to and including the P_2-term):

$$f(x) = \begin{cases} 0, & -1 < x < 0, \\ x, & 0 < x < 1. \end{cases}$$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$$\int_{-1}^{1} f(x) P_\ell(x)\, dx = \frac{2}{2 \cdot \ell + 1} c_\ell$$

$$\Rightarrow \quad c_0 = \frac{1}{2}\int_{-1}^{1} f(x)\, dx = \frac{1}{2}\int_0^1 x\, dx = \frac{1}{4}$$

$$c_1 = \frac{3}{2}\int_0^1 x^2\, dx = \frac{1}{2}$$

$$c_2 = \frac{5}{2}\int_0^1 \frac{1}{2}(3x^3 - x)\, dx = \frac{5}{4}\left(\frac{3}{4} - \frac{1}{2}\right) = \frac{5}{16}$$

$$\underline{\underline{f(x) = \frac{1}{4} P_0(x) + \frac{1}{2} P_1(x) + \frac{5}{16} P_2(x) + \cdots}}$$

5. (4 points) Use the recurrence relation

$$lP_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x)$$

to show that

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}.$$

$$2n\, P_{2n}(0) = 0 - (2n-1)\, P_{2n-2}(0)$$

$$P_{2n}(0) = -\frac{2n-1}{2n} P_{2n-2}(0) = +\frac{(2n-1)(2n-3)}{(2n)(2n-2)} P_{2n-4}(0)$$

$$= \cdots = (-1)^n \frac{(2n-1)(2n-3)\cdots(1)}{(2n)(2n-2)\cdots(2)} \cdot P_0(0)$$

$$= (-1)^n \frac{(2n)!}{[2n(2n-2)\cdots(2)]^2} = (-1)^n \frac{(2n)!}{[2^n \cdot n!]^2}$$

$$= (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$$