30.2 Fall 2017 Midterm 2

1. For each matrix below, write the expression for e^{At}

   (a)
   $$A_1 = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}, \qquad e^{A_1 t} =$$

   (b)
   $$A_2 = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}, \qquad e^{A_2 t} =$$

   (c)
   $$A_3 = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}, \qquad e^{A_3 t} =$$

   (d)
   $$A_4 = \begin{bmatrix} -2+j5 & 0 \\ 0 & -2-j5 \end{bmatrix}, \qquad e^{A_4 t} =$$

   (e)
   $$A_5 = \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}, \qquad e^{A_5 t} =$$

2. Consider the quadratic polynomial $p(s)$, which depends on two real-valued parameters β_1 and β_2,
   $$p(s) = s^2 - s + 1 + \beta_1(7s+4) + \beta_2(5s+3)$$
   Find the values of β_1 and β_2 so that the roots of $p(s)$ are at $\{-2+j1, -2-j1\}$. **Hint:** What quadratic polynomial has roots at $\{-2+j1, -2-j1\}$

3. When we studied studied how Simulink worked, we saw that in order to simulate an interconnection of dynamical systems, the code simply needed to "call" each individual system, often in a specific order, in order to determine the entire state-derivative, and then do this repeatedly to compute an approximate, numerical solution to the ODEs. This strategy of keeping all of the systems separate provides generality that is especially useful when simulating interconnections of systems that are nonlinear. For interconnections of linear systems (governed by state equations), we can often **explicitly determine the state equations of the interconnection, since all of the necessary substitutions are simple (because of linearity)**. That is the task in this problem.

   Suppose the plant P is described by

$$\mathbf{P}: \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Ed(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array}$$

   where A, E, B, C are matrices with dimensions

$$A \in \mathbb{R}^{n\times n}, \quad E \in \mathbb{R}^{n\times v}, \quad B \in \mathbb{R}^{n\times m}, \quad C \in \mathbb{R}^{q\times n}$$

   and the signals are of dimensions

$$x(t) \in \mathbb{R}^{n}, \quad d(t) \in \mathbb{R}^{v}, \quad u(t) \in \mathbb{R}^{m}, \quad y(t) \in \mathbb{R}^{q}$$

   Suppose the controller is also described by a linear system model, namely

$$\mathbf{C}: \begin{array}{rcl} \dot{z}(t) & = & Fz(t) + Gr(t) + Hy_m(t) \\ u(t) & = & Jz(t) + Kr(t) + Ly_m(t) \end{array}$$

   where F, G, H, J, K, L are matrices of dimension

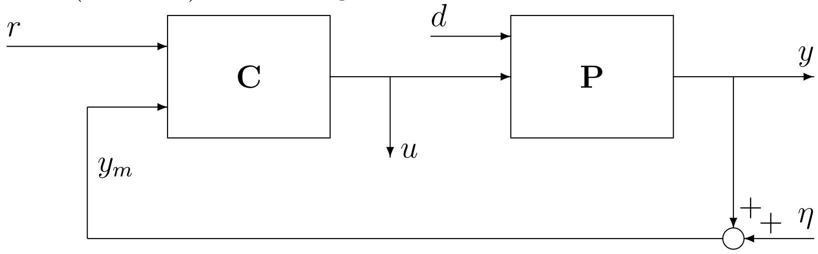
$$F \in \mathbb{R}^{w\times w}, \quad G \in \mathbb{R}^{w\times f}, \quad H \in \mathbb{R}^{w\times q}, \quad J \in \mathbb{R}^{m\times w}, \quad K \in \mathbb{R}^{m\times f}, \quad L \in \mathbb{R}^{m\times q},$$

   and the signals are of dimensions

$$z(t) \in \mathbb{R}^{w}, \quad r(t) \in \mathbb{R}^{f}$$

   The simple model for sensor-noise is $y_m(t) = y(t) + \eta(t)$ where $\eta(t) \in \mathbb{R}^q$.

   The (familiar) block diagram is:



   The task in this problem is to find the state-equation model for the closed-loop system, with

$$\text{inputs} = \begin{bmatrix} r \\ d \\ \eta \end{bmatrix}, \qquad \text{states} = \begin{bmatrix} x \\ z \end{bmatrix}, \qquad \text{outputs} = \begin{bmatrix} y \\ u \end{bmatrix}$$

Task/Question: Fill in the "'block" 4×5 matrix the correctly describes the closed-loop system.

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \\ y(t) \\ u(t) \end{bmatrix} = \left[\begin{array}{c|c|c|c|c} & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \end{array}\right] \begin{bmatrix} x(t) \\ z(t) \\ r(t) \\ d(t) \\ \eta(t) \end{bmatrix}$$

Make sure that your matrix products are in the correct order (nothing is scalar here, so you should not be sloppy about order). If you have time, convince yourself that all the dimensions make sense! That will also help you find errors...

4. Suppose $A \in \mathbb{R}^{3\times 3}$ and $AV = V\Lambda$, where

$$V = \begin{bmatrix} -1+j4 & -1-j4 & 0 \\ 2-j1 & 2+j1 & 1 \\ 6 & 6 & -1 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -2+j2 & 0 & 0 \\ 0 & -2-j2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Find matrices $W \in \mathbb{R}^{3\times 3}$ and $\Gamma \in \mathbb{R}^{3\times 3}$ (note - these are real-valued matrices, in contrast to V and Λ, which are complex) such that

$$AW = W\Gamma$$

where W is invertible, and Γ is "block-diagonal". **Note: You do not have to prove that W is invertible, but whatever you write down should be invertible.**

5. A one-state plant, P is governed by equations

$$\dot{x}(t) = Ax(t) + B_1 d(t) + B_2 u(t), \qquad y(t) = Cx(t)$$

where x is the state of the plant, d is an external disturbance, and u is the control variable. The plant output is y. The constants A, B_1, B_2, C are referred to as the plant parameters, and are assumed known, with $B_2 \neq 0$ and $C \neq 0$.

A feedback control system is proposed, which uses the reference input r and measures y (no measurement noise for this problem, to keep the notation to a minimum) to produce u. The goal of control is

Goal1: closed-loop should be stable

Goal2: the eigenvalues of the closed-loop system can be assigned to desired values by appropriate choices of the parameters within the controller's equations.

Goal3: steady-state gain from $r \to y$ should equal 1

Goal4: steady-state gain from $d \to y$ should equal 0

Goal5: the objective in **Goal3** should be robust to "modest" changes in the plant parameters. Obviously, if **Goal3** is unachievable, then **Goal5** is also unachievable.

Goal6: the objective in **Goal4** should be robust to "modest" changes in the plant parameters. Obviously, if **Goal4** is unachievable, then **Goal6** is also unachievable.

(a) Consider a proportional controller of the form

$$u(t) = K_P(r(t) - y(t))$$

Which goals are achievable (by proper choice of K_P), and which goals are unachievable (regardless of the choice)? **Hint:** if you are unsure about acheiving **Goal1** and/or **Goal2** for any of these problems, consider the plans $\dot{x}(t) = x(t) + u(t), y(t) = x(t)$, which is a simple unstable plant on which you can gain insight.

(b) Consider a proportional controller of the form

$$u(t) = K_1 r(t) + K_2 y(t)$$

Which goals are achievable (by proper choice of K_1 and K_2), and which goals are unachievable (regardless of the choice)?

(c) Consider an integral controller of the form

$$\dot{q}(t) = r(t) - y(t), \qquad u(t) = K_I q(t)$$

Which goals are achievable (by proper choice of K_I), and which goals are unachievable (regardless of the choice)?

(d) Consider a proportional/integral controller of the form

$$\dot{q}(t) = r(t) - y(t), \qquad u(t) = K_I q(t) + K_P(r(t) - y(t))$$

Which goals are achievable (by proper choice of K_I and K_P), and which goals are unachievable (regardless of the choice)?

6. A one-state plant, P is governed by equations

$$\dot{x}(t) = -4x(t) + d(t) + 3u(t), \qquad y(t) = 2x(t)$$

where x is the state of the plant, d is an external disturbance, and u is the control variable. The plant output is y. A reference input r is available to the controller.

(a) Design a PI controller of the form

$$\dot{q}(t) = r(t) - y(t), \qquad u(t) = K_I q(t) + K_P(r(t) - y(t))$$

such that the closed-loop eigenvalues are given by $(\xi = 0.9, \omega_n = 10)$.

(b) In the closed-loop system, what is the steady-state gain from $r \rightarrow y$?

(c) In the closed-loop system, what is the steady-state gain from $d \rightarrow y$?

(d) In the closed-loop system, what is the steady-state gain from $d \rightarrow u$?