Omer Braude

good!

50/50 ☆

First Midterm Exam

Place all answers on the question sheet provided. The exam is closed textbook/notes/handouts/homework. You are allowed to use a calculator, but not a computer, tablet or smartphone, as well as 2 double-sided formula sheets. Write all answers clearly and in complete sentences. All answers should be supported by analysis or an argument. This exam has a total of 50 points.

1. Ben owns a store where among other things he sells "spicy lollipops". He has one lollipop stand on the check-out counter that fits up to 5 lollipops. However, the "spicy lollipops" don't sell very well, and in Ben's experience, he sells on any given day either no lollipops, with probability 0.6, one lollipop, with probability 0.3, or two lollipops, with probability 0.1. Let X_n denote the number of "spicy lollipops" on the stand at the end of day n, right after the store has closed for the day. If at this point Ben sees that the stand is empty we will refill it with 5 new "spicy lollipops". Note that he will not refill the stand while the store is open, and any unmet demand is lost.

   (a) [8 PTS] Write the one-step transition probability matrix.

\# of lollipops

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix}
0, & 0, & 0, & 0.1, & 0.3, & 0.6 \\
0.4, & 0.6, & 0, & 0, & 0, & 0 \\
0.1, & 0.3, & 0.6, & 0, & 0, & 0 \\
0, & 0.1, & 0.3, & 0.6 & 0 & 0 \\
0, & 0, & 0.1, & 0.3, & 0.6, & 0 \\
0, & 0, & 0, & 0.1, & 0.3, & 0.6
\end{bmatrix}
\end{array}
$$

I used commas to make a clearer distinction between entries.

Note: For cases where there is unmet demand I consider the prob of having X as the prob of having anything less than or equal to X

(b) [4 PTS] The stationary distribution of $\{X_n : n \geq 1\}$ is given by:

$$(\pi_0, \pi_1, \pi_2, \pi_4, \pi_4, \pi_5) = (0.0962, 0.1925, 0.1915, 0.1953, 0.1803, 0.1442).$$

How often on average does Ben have to refill the stand? You may assume that he does not refill it until it is empty, and only after the store has closed.

Since he fills it on the days that the shelf is empty, it happens 0.0962 of the days. If the store is open every day, then he refills the stand every $\frac{1}{0.0962} \approx 10.4$ days.

(c) [8 PTS] The "spicy lollipop" stand currently has 2 lollipops, and the store has closed for the day. Compute the expected number of days until Ben has to refill it.

I can make a new one step matrix:

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.1 & 0.3 & 0.6 \\ 1 & 0.4 & 0.6 & 0 \\ 2 & 0.1 & 0.3 & 0.6 \end{array}$$

If I can compute how often he refills the stand in this scenario. It will be the same as the expected time that the problem asks for, because coming back to 0 from 0 is the same as going from 2 to 0.

$$\begin{cases} [\pi_0 \ \pi_1 \ \pi_2] \begin{bmatrix} .1 & .3 & .6 \\ .4 & .6 & 0 \\ 0.1 & .3 & .6 \end{bmatrix} = [\pi_0 \ \pi_1 \ \pi_2] \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\begin{cases} 0.1\pi_0 + 0.4\pi_1 + 0.1\pi_2 = \pi_0 \\ 0.6\pi_0 + 0.6\pi_2 = \pi_2 \rightarrow 3\pi_0 = 2\pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\rightarrow \begin{cases} \pi_0 + \pi_1 + \frac{3}{2}\pi_0 = 1 \rightarrow \pi_1 = 1 - \frac{5}{2}\pi_0 \\ 0.1\pi_0 + 0.4\pi_1 + 0.15\pi_0 = \pi_0 \end{cases}$$

$$-0.75\pi_0 + 0.4 - \pi_0 = 0 \Rightarrow 0.4 = 1.75\pi_0$$

$$\pi_0 = 0.22857$$

$$\frac{1}{0.22857} = \boxed{4.375 = E[\text{days until he refills}]}$$

2. Let $\{X_n : n \geq 1\}$ be a Markov chain on the states $\{0, 1, 2, 3, 4\}$ having one-step transition probabilities:

$$P = \begin{bmatrix} 1-p & p & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1-p & p \end{bmatrix}$$

with $p \in (0, 1)$.

(a) [4 PTS] Is the Markov chain time-reversible? Explain.

It is time reversible, becasc it can be categorized as birth-death markov chain. It may only move by 1 at a time so we know its time reversible.

(b) [4 PTS] Is this Markov chain aperiodic? Explain.

Yes, each one of the states can be visited at any number of steps once we reach a high number of steps. We can also say that the states have period 1.

For example any state X_i can be vistited after $n, 2n, 3n \ldots n \cdot n$ steps for large enough n, regardless of where it has started

(c) [8 PTS] For the case when $p = 1/3$ set up a system of linear equations that when solved gives the stationary distribution of $\{X_n : n \geq 1\}$. Do not solve it.

$$\begin{cases} \vec{\pi} \cdot \vec{P} = \vec{\pi} \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

$$\begin{cases} \pi_0 \cdot \frac{2}{3} + \pi_1 \frac{2}{3} = \pi_0 \\ \pi_0 \cdot \frac{1}{3} + \pi_2 \frac{2}{3} = \pi_1 \\ \pi_1 \cdot \frac{1}{3} + \pi_3 \cdot \frac{2}{3} = \pi_2 \\ \pi_3 \cdot \frac{1}{3} + \pi_4 \cdot \frac{1}{3} = \pi_4 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

I dropped the equation corresponding to 4th collumn of P.

(d) [4 PTS] The stationary distribution for the case when $p = 1/4$ is given by:

$$(\pi_0, \pi_1, \pi_2, \pi_3, \pi_4) = (0.6694, 0.2231, 0.0744, 0.0248, 0.0083)$$

In the long run, what proportion of time is X_n odd?

The π_i that correspond to odd numbers are:

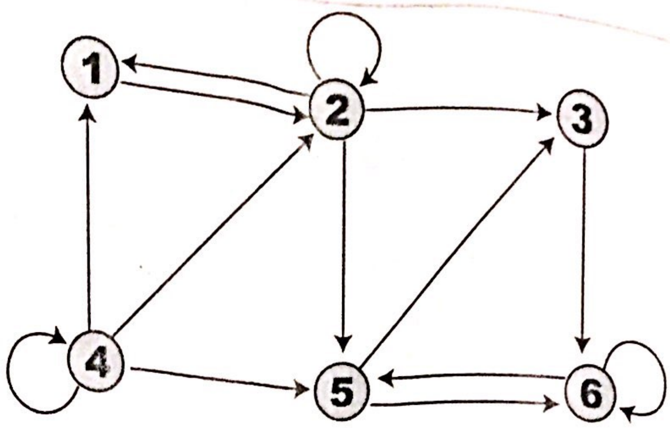
π_1 and π_3

$\pi_1 + \pi_3 = 0.2231 + 0.0248 = \boxed{0.2479}$

Proportion X_n is odd for long run.

Because it's aperiodic and irreducable I can use this method.

3. Consider a random walk on the following directed graph:



Whenever at vertex i, $i \in \{1,2,3,4,5,6\}$ the next vertex is chosen uniformly at random among all outbound arcs, i.e., if there are 4 outgoing arcs, then each of the four arcs is chosen with probability 1/4. Let X_n denote the vertex where the random walk is after n steps.

(a) [5 PTS] Identify the transient and recurrent states.

4, 1 and 2 are transient, since once we leave them, we may never get back.

3, 5, 6 are recurrent, because they are not transient. We know for certain that will will visit them after infinite steps.

(b) [5 PTS] Compute the expected number of times the random walk will be on vertex 4 given that $X_0 = 4$; do not include the initial position.

Can be computed by treating this number X as a geometric r.v. with probability $\frac{3}{4}$.

$\underbrace{(\text{sum } P(4,1), P(4,5), P(4,2))}$

The $E[X] = \frac{1}{p} = \frac{1}{3/4} = \frac{4}{3}$. That means that on average it will take $\frac{4}{3}$ steps to leave 4 and never get back to it. In other words, every 3 trials we will have one trial in which we visit 4 again (on average)

So the answer is $\boxed{\frac{1}{3}}$

another way: Let X be # of visits in 4:

$E[X] = \frac{1}{4}(E[X]+1) \rightarrow E[X] = \frac{1}{3}$