Math 1A (Fall 2017) Midterm I (Thursday September 14, 3:40-5:00)

1. Mark each of the following True (T) or False (F). No justification is necessary. (For each sub-problem, correct = 4 pts, no response = 2 pts, wrong = 0 pts.)

(1) (F) If f is an odd function and g is an even function then the composite function $f \circ g$ is odd.

In fact $f \circ g$ will always be even (and will only be odd if $f \circ g = 0$). To see this, observe that $g(-x) = g(x)$ since g is even, so

$$(f \circ g)(-x) = f(g(-x)) = f(g(x)) = (f \circ g)(x).$$

(2) (T) If f is a one-to-one function defined on $\mathbb{R}$ then f^{-1} is also a one-to-one function.

This is true because f^{-1} has an inverse, namely f. Alternatively, for f^{-1} to be one-to-one, it has to pass the horizontal line test, which is equivalent to f passing the vertical line test. This always happens because f is a function.

(3) (F) If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$. Then $\lim_{x\to a} f(x)g(x)$ is either ∞ or 0.

For example, $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$ and $\lim_{x\to 0} x^2 = 0$, but $\lim_{x\to 0} \dfrac{1}{x^2} \cdot x^2 = \lim_{x\to 0} 1 = 1$

(4) (T) The line $x = 1$ is a vertical asymptote of $y = \dfrac{1}{x-1}$ because $\lim_{x\to 1^-} \dfrac{1}{x-1} = -\infty$.

It's true that $x = 1$ is a vertical asymptote since $\lim_{x\to 1^-} \dfrac{1}{x-1}$ is indeed $-\infty$. One can see this by plotting the graph $y = \dfrac{1}{x-1}$. (One way to see that it's $-\infty$, not ∞ is: $x \to 1^-$ means in particular that $x < 1$, so $x - 1$ and thus also $\frac{1}{x-1}$ are negative-valued.)

(5) (F) If f is defined on $(0, \infty)$ and $f(\frac{1}{n}) = 0$ for $n = 1, 2, 3, \ldots$ then $\lim_{x\to 0^+} f(x) = 0$.

The condition is not sufficient for the limit to be equal to 0. You need that the values of $f(x)$ should approach 0 for *all* (for instance irrational) values of x approaching 0. For a concrete example, $f(x) = \sin(\pi/x)$ is zero for $x = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ but $\lim_{x\to 0^+} f(x)$ does not exist.

2. (15 pts) Answer the following.
(1) (5 pts) Compute $\sin(\tan^{-1}(2))$.

Solution 1: If you put $y = \tan^{-1}(2)$ then $\tan y = 2$ and $0 < y < \pi/2$ so this can be modeled by a right triangle with sides $1, 2, \sqrt{5}$ such that y is the angle between sides $\sqrt{5}$ and 1. Therefore $\sin(\tan^{-1}(2)) = \sin(y) = \boxed{2/\sqrt{5}}$.

Solution 2: Again, with $y = \tan^{-1}(2)$, you know $\tan y = 2$ and $0 < y < \pi/2$ so $\sin(\tan^{-1}(2)) = \sin y > 0$. Now recall

$$\sin^2 y + \cos^2 y = 1.$$

Dividing by $\sin^2 y$, get $1 + \dfrac{1}{\tan^2 y} = \dfrac{1}{\sin^2 y}$. Plugging in $\tan y = 2$, get

$$1 + \frac{1}{4} = \frac{1}{\sin^2 y}, \quad \text{so} \quad \sin^2 y = \frac{4}{5}.$$

Since $\sin y > 0$, we have $\sin y = \sqrt{4/5} = 2/\sqrt{5}$.

(Note: In fact the argumet of either solution shows that $\sin(\tan^{-1}(x)) = \dfrac{x}{\sqrt{x^2+1}}$.)

(2) (10 pts) Let $f(x) = \ln(x-1)$ and $g(x) = \dfrac{3x+1}{x}$. What is the domain of the composite function $g \circ f$?

Solution: The domain of f is $(1, \infty)$ and the domain of g is $\mathbb{R} - \{0\}$. So the domain of $f \circ g$ consists of numbers satisfying the two conditions:

- x is in the domain of f, namely $x > 1$.
- $\ln(x-1)$ is in the domain of g, so $\ln(x-1) \neq 0$, namely $x - 1 \neq 1$. So the condition is $x \neq 2$.

Therefore the answer is the set of numbers $x > 1$ such that $x \neq 2$. Namely the answer is $\boxed{(1,2) \cup (2, \infty)}$.

3. (15 pts) Let $f(x) = \begin{cases} |x|/x^3, & x < 0, \\ \sin(1/x), & x > 0. \end{cases}$ Find $\lim_{x \to 0^-} x^2 f(x)$ and $\lim_{x \to 0^+} x^2 f(x)$.
For either limit, if the limit does not exist, explain why.

Solution:

For the limit as $x \to 0^-$, we have $x < 0$, so $|x| = -x$ and $f(x) = |x|/x^3 = -1/x^2$. Therefore

$$\lim_{x \to 0^-} x^2(-1/x^2) = \lim_{x \to 0^-} (-1) = \boxed{-1}.$$

Notice that $-1 \leq \sin(1/x^2) \leq 1$. Multiplying x^2, which is positive for $x > 0$, we obtain

$$-x^2 \leq x^2 \sin(1/x) \leq x^2.$$

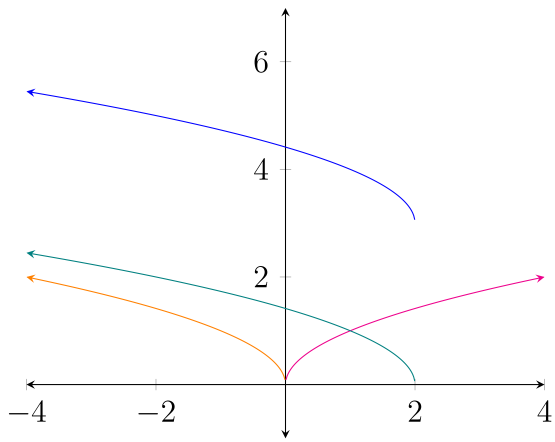
Since $\lim_{x \to 0^+}(-x^2) = \lim_{x \to 0} x^2 = 0$, the squeeze theorem implies that

$$\lim_{x \to 0^+} x^2 \sin(1/x) = \boxed{0}.$$

4. (15 pts) Explain how the graph of the function $k(x) = \sqrt{2-x} + 3$ is obtained from the graph $y = \sqrt{x}$ by a sequence of basic transformations (shifting, expanding, shrinking, or reflecting the graph). Make sure to indicate in what order the transformations are performed. Using this, sketch the graph of $\sqrt{2-x} + 3$ in the xy-plane.

Solution: This function comes from a sequence of transformations of $f(x) = \sqrt{x}$. To graph it, first try drawing the graph of $f(x)$ (in pink) to remind yourself of the shape of the square root graph. Then try graphing $g(x) = \sqrt{-x}$ (in orange). This is a reflection over the y axis. Next graph $h(x) = \sqrt{2-x}$ (in green). This is a shift to the right by 2. Finally, graph the original function $k(x)$ (in blue). This is $h(x)$ shifted up by 3.

(Note: The answer is not unique. As long as your sequence of transformations does give $y = \sqrt{2-x} + 3$, your answer is considered correct.)



5. (15 pts) Consider the function $f(x) = \ln(2 + e^x)$ on $\mathbb{R}$, which is one-to-one. Find the formula for $f^{-1}(x)$.

Solution:

We solve $y = \ln(2 + e^x)$ for x. By taking exponential, we have

$$e^y = 2 + e^x,$$

so $e^y - 2 = e^x$. Taking ln we get

$$x = \ln(e^y - 2).$$

Thus the answer is $f^{-1}(y) = \ln(e^y - 2)$, or $\boxed{f^{-1}(x) = \ln(e^x - 2)}$.

6. (20 pts) Compute the following limits if they exist. For either limit, if the limit does not exist, explain why.

(1) $\lim_{x\to 1}\left(\frac{1}{1-x} - \frac{3}{1-x^3}\right)$

Solution:

$$\lim_{x\to 1}\left(\frac{1}{1-x} - \frac{3}{1-x^3}\right) = \lim_{x\to 1}\left(\frac{1+x+x^2}{(1-x)(1+x+x^2)} - \frac{3}{(1-x)(1+x+x^2)}\right)$$

$$= \lim_{x\to 1}\frac{x^2+x-2}{(1-x)(1+x+x^2)} = \lim_{x\to 1}\frac{(x+2)(x-1)}{(1-x)(1+x+x^2)}$$

$$= \lim_{x\to 1}\frac{-(x+2)}{(1+x+x^2)} = \frac{-3}{3} = \boxed{-1}.$$

(2) $\lim_{t\to 0}\left(\frac{\sqrt{1+t}-\sqrt{1-t}}{t}\right)$

Solution:

$$\lim_{t\to 0}\left(\frac{\sqrt{1+t}-\sqrt{1-t}}{t}\right) = \lim_{t\to 0}\left(\frac{(\sqrt{1+t}-\sqrt{1-t})(\sqrt{1+t}+\sqrt{1-t})}{t(\sqrt{1+t}+\sqrt{1-t})}\right)$$

$$= \lim_{t\to 0}\left(\frac{(1+t)-(1-t)}{t(\sqrt{1+t}+\sqrt{1-t})}\right) = \lim_{t\to 0}\left(\frac{2t}{t(\sqrt{1+t}+\sqrt{1-t})}\right)$$

$$= \lim_{t\to 0}\left(\frac{2}{\sqrt{1+t}+\sqrt{1-t}}\right) = \frac{2}{1+1} = \boxed{1}.$$