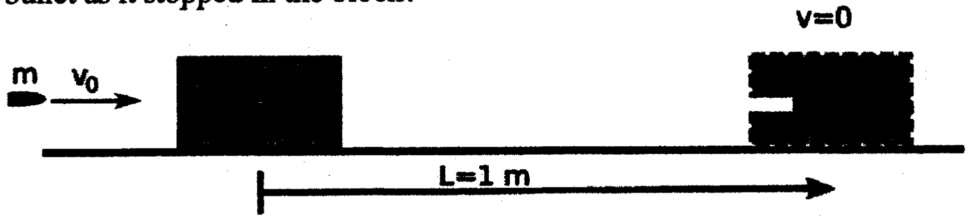
1. A rectangular block of mass M = 500 g (grams) rests on a horizontal surface. The coefficient of kinetic friction between the block and the surface is $\mu_k = 0.2$. A bullet of mass m = 2 g fired horizontally strikes the block. The block slides a distance L = 1 m before stopping.

a. (10 pts) Find the speed of the bullet before it struck the block

b. (5 pts) Find the fraction of the bullet's original kinetic energy that is lost as internal energy.

c. (5 pts) If the bullet traveled 2 cm into the block before stopping, what was the average force (in Newtons) exerted on the bullet as it stopped in the block?



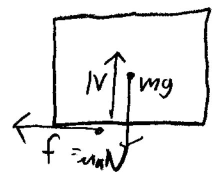
a) This is a collision, so we use conservation of momentum

$$mv_0 = (m+M)v_c \quad , \quad V_c = V_0\left(\frac{m}{m+M}\right)$$

next, the kinetic friction will apply a constant force, causing a constant accelleration

$$F = \mu_k N = \mu_k mg \quad , \quad a = \mu_k g$$

We know that given a constant acceleration



$$V_f^2 - V_0^2 = 2ax$$

$$-V_c^2 = -2ax$$

$$x = \frac{V_c^2}{2a} = V_0^2\left(\frac{m}{m+M}\right)^2\left(\frac{1}{2\mu_k g}\right)$$

now, we can solve this for V_0

$$2\mu_k g x\left(\frac{m+M}{m}\right)^2 = V_0^2 \quad , \quad \boxed{V_0 = \frac{m+M}{m}(2\mu_k g x)^{\frac{1}{2}}} = \frac{502}{2}(2(.2)\cdot 10\cdot 1)^{\frac{1}{2}} = \boxed{502\ \tfrac{m}{s}}$$

b) $E_0 = \frac{1}{2}mv_0^2$

$$E_f = \frac{1}{2}(m+M)V_c^2 = \frac{1}{2}(m+M)V_0^2\left(\frac{m}{m+M}\right)^2 = \frac{1}{2}V_0^2\left(\frac{m^2}{m+M}\right)$$

$$\Delta E = E_0 - E_f = \frac{1}{2}V_0^2\left(m - \frac{m^2}{m+M}\right) = \frac{1}{2}V_0^2\left(\frac{m^2 + mM - m^2}{m+M}\right) = \frac{1}{2}V_0^2\left(\frac{mM}{m+M}\right)$$

$$\frac{\Delta E}{E_0} = \frac{\frac{1}{2}V_0^2\left(\frac{mM}{m+M}\right)}{\frac{1}{2}V_0^2 m} = \boxed{\frac{M}{m+M}} = \frac{500}{502} = \boxed{.996}$$

$E_f = 0$ is also acceptable as the final energy after the block stops is 0

c) We will use $W = F_{avg} d$

$$W = \Delta E \ , \quad E_0 = \frac{1}{2}mV_0^2 \quad E_f = \frac{1}{2}mV_c^2 = \frac{1}{2}V_0^2 m\left(\frac{m}{m+M}\right)^2 = \frac{1}{2}V_0\frac{m^3}{(m+M)^2}$$

$$\Delta E = \frac{1}{2}mV_0^2 - \frac{1}{2}mV_0^2\left(\frac{m}{m+M}\right)^2 = \frac{1}{2}mV_0^2\left(1 - \left(\frac{m}{m+M}\right)^2\right)$$

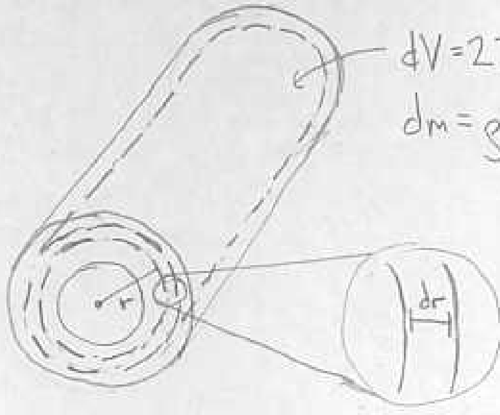
$$F_{avg} = \frac{W}{d} = \frac{\Delta E}{d} = \boxed{\frac{\frac{1}{2}mV_0^2\left(1 - \left(\frac{m}{m+M}\right)^2\right)}{d}} = \frac{.5(.002)502^2\left(1 - \left(\frac{2}{502}\right)^2\right)}{.02} = \boxed{1.26\times 10^4\ N}$$

$\boxed{2}$

a) The Volume of the pipe is the volume of a cylinder with radius R_2 $(\pi R_2^2 \ell)$ minus the volume of a cylinder with radius R_1 $(\pi R_1^2 \ell)$

i.e. $\pi \ell (R_2^2 - R_1^2)$

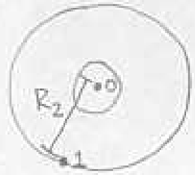
$$M = \rho V = \boxed{\pi \ell \rho (R_2^2 - R_1^2) = M}$$



$dV = 2\pi r \ell dr$

$dm = \rho dV = 2\pi \ell \rho r dr$

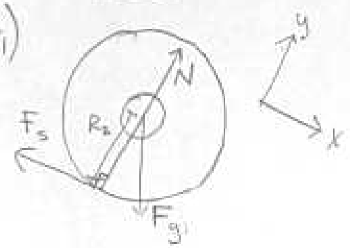
$$I_0 = \int_{R_1}^{R_2} r^2 dm = \frac{\pi \ell \rho}{2}(R_2^4 - R_1^4) = \boxed{\frac{M}{2}(R_1^2 + R_2^2)}$$



By the parallel axis theorem,

$$I_1 = I_0 + MR^2 = \boxed{\frac{M}{2}(R_1^2 + 3R_2^2) = I_1}$$

b) i)



$F_{gx} = mg\sin\theta$

$F_{gy} = mg\cos\theta$

$$\left.\begin{aligned} &\sum F_x = ma_x : \; mg\sin\theta - f_s = ma_{CM} \\ &\sum \tau_0 = I_0\alpha : \; R_2 f_s = I_0 \alpha \\ &a_{CM} = \alpha R_2 \end{aligned}\right\} \rightarrow \boxed{a_{CM} = \frac{2gR_2^2\sin\theta}{3R_2^2 + R_1^2}}$$

ii) $$\alpha = \frac{a_{CM}}{R_2} = \boxed{\frac{2gR_2\sin\theta}{3R_2^2 + R_1^2} = \alpha}$$

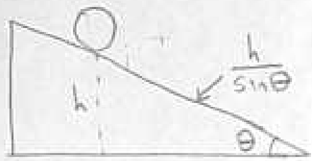
iii) $$R_2 f_s = I_0\alpha \rightarrow f_s = \frac{I_0\alpha}{R_2} = \boxed{Mg\frac{R_2^2 + R_1^2}{3R_2^2 + R_1^2}\sin\theta = f_s}$$

iv) $$\left.\begin{aligned} &N = mg\cos\theta \\ &f_s = \mu N \end{aligned}\right\} \rightarrow \mu = \frac{f_s}{mg\cos\theta} = \boxed{\frac{R_2^2 + R_1^2}{3R_2^2 + R_1^2}\tan\theta = \mu}$$

when μ is minimal

2, cont'd

b v)



$$V_{CM}^2 = \cancel{v_i^2}^{\,0} + 2a_{CM}\frac{h}{\sin\theta} \rightarrow \boxed{V_{CM} = \sqrt{\frac{4ghR_2^2}{3R_2^2+R_1^2}}}$$

c) $\boxed{W_g = Mgh}$ always

$\boxed{W_N = 0}$ b/c $\vec{N} \perp$ motion

Since $\vec{f}_s$ is in the opposite direction of motion,

$$W_{Fr,trans} = -f_s\frac{h}{\sin\theta} = \boxed{-Mgh\frac{R_2^2+R_1^2}{3R_2^2+R_1^2} = W_{Fr,trans}}$$

The pipe travels through a total angle of $\frac{h}{\sin\theta}\frac{1}{R_2}$

(since, along the edge of the pipe, arclength $= R_2\Delta\theta$) so

$$W_{Fr,rot} = \int_{\theta_i}^{\theta_f}\tau\, d\theta = \int_0^{\frac{h}{\sin\theta R_2}} f_s R_2\, d\theta = f_s \cancel{R_2}\frac{h}{\sin\theta}\frac{1}{\cancel{R_2}} = \boxed{Mgh\frac{R_2^2+R_1^2}{3R_2^2+R_1^2} = W_{Fr,rot}}$$

So $W_{Fr,total} = 0$. This makes sense since the force of friction acts over no distance, since the point of contact doesn't move.

$$KE_{trans,F} = \frac{1}{2}Mv_{CM}^2 = \boxed{\frac{2R_2^2}{3R_2^2+R_1^2}Mgh = KE_{trans}}$$

$$KE_{rot,F} = \frac{1}{2}I_0\omega^2 = \frac{1}{2}I_0\frac{V_{CM}^2}{R_2^2} = \boxed{\frac{R_1^2+R_2^2}{3R_2^2+R_1^2}Mgh = KE_{rot}}$$ ($\omega = \frac{v}{R}$: rolling condition)

$$\cancel{KE_i}^{\,0} + W_g + \cancel{W_N}^{\,0} + \cancel{W_{Fr,total}}^{\,0} \overset{?}{=} KE_{trans} + KE_{rot}$$

$$Mgh \overset{?}{=} \frac{2R_2^2}{3R_2^2+R_1^2}Mgh + \frac{R_1^2+R_2^2}{3R_2^2+R_1^2}Mgh$$

$$Mgh = Mgh \checkmark$$

$\boxed{\text{Energy is conserved}}$

$\boxed{3}$

a) Momentum conservation: $(2.0\,\text{kg})(8.0\,\text{m/s}) = (2.0\,\text{kg} + 4.5\,\text{kg})v$

$v = 2.46\,\text{m/s}$

840 N/m

Energy conservation: $\frac{1}{2}(2.0\,\text{kg})(8.0\,\text{m/s})^2 = \frac{1}{2}(2.0\,\text{kg} + 4.5\,\text{kg})v^2 + \frac{1}{2}kx^2$

$\boxed{x = .32\,\text{m}}$

Rubric

Using momentum conservation: { Incorrectly +1; correctly +3 }

" energy " : { Incorrectly +1; correctly +3 }

" hint : +2

correct answer : +2

Saying $v=0$ and doing calculations correctly: +4

" " " " " incorrectly: +3

b) Initial: Final:

Momentum conservation: $(2.0\,\text{kg})(8.0\,\text{m/s}) = (2.0\,\text{kg})v_1 + (4.5\,\text{kg})v_2$ (1)

Energy conservation: $\cancel{\tfrac{1}{2}}(2.\cancel{0}\,\text{kg})(8.0\,\text{m/s})^2 = \cancel{\tfrac{1}{2}}(2.\cancel{0}\,\text{kg})v_1^2 + \tfrac{1}{2}(4.5\,\text{kg})v_2^2$ (2)

Divide (1) by 2, square it, subtract off (2), and solve for v_2: (3)

$v_2 = -1.6v_1$

Plug into (1) and solve for v_1:

$\boxed{v_1 = -3.077\,\text{m/s}}$

Plug into (3):

$\boxed{v_2 = 4.923\,\text{m/s}}$

$\boxed{\text{The collision is elastic because the spring force is conservative}}$

3b

Rubric

Using momentum conservation: { incorrectly: +1 ; correctly: +2 }

" energy " : { incorrectly: +1 ; correctly: +2 }

"Collision is elastic" +1

" " " " b/c energy is conserved" +2

" " " " b/c spring force is conservative" +3

Correct answers: +3

3b

Rubric

Using momentum conservation: { incorrectly: +1
correctly: +2 }

" energy " : { incorrectly: +1
correctly: +2 }

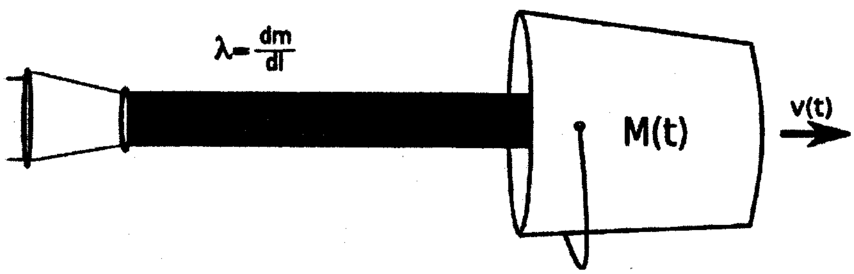
{ "Collision is elastic" +1
" " " " b/c energy is conserved" +2
" " " " b/c spring force is conservative" +3 }

Correct answers: +3

4. An astronaut in the space shuttle (no gravity) squirts a steady stream of sand, with constant mass per unit length λ, into a bucket with velocity V_s. Assume the sand collides inelastically with the bucket and is collected in the bucket.

a) (10 points) Find the equation of motion for the bucket, expressing the acceleration of the bucket, dv/dt, in terms of M(t), V_s, and the mass of sand striking the bucket per unit time, dm/dt.

b) (5 points) Derive an expression for dm/dt in terms of λ, v(t), and V_s.



a) We will use $F=ma$, using the momentum transfer of the stream to find our F

$F = \frac{dp}{dt} \simeq \frac{\Delta p}{\Delta t}$

lets consider some small period of time Δt, during which $V(t)$, $M(t)$ and $\frac{dm}{dt}$ are constant

during this time we want to find $\Delta \bar{p}$

$\bar{p} = m\bar{v}$, so $\Delta \bar{p} = $(mass of sand that strikes the bucket)(change in velocity of that sand)

by assumption, $\frac{dm}{dt}$ is constant during this time, so $m = \frac{dm}{dt} \cdot \Delta t$ (total mass) = (rate of mass)(time)

each piece of sand then changes velocity from V_s to $V(t)$

Thus: $\Delta \bar{p} = \left(\frac{dm}{dt}\Delta t\right)\left(\bar{V}_s - \bar{V}(t)\right)$, $\frac{\Delta \bar{p}}{\Delta t} = \frac{\left(\frac{dm}{dt}\Delta t\right)\left(\bar{V}_s - \bar{V}(t)\right)}{\Delta t} = \left(\bar{V}_s - \bar{V}(t)\right)\frac{dm}{dt} = F$

So $\bar{a} = \frac{d\bar{v}}{dt} = \frac{\bar{F}}{M(t)} = \boxed{\left(\frac{\bar{V}_s - \bar{V}(t)}{M(t)}\right)\frac{dm}{dt} = \frac{d\bar{v}}{dt}}$

b) again, lets consider some small period of time Δt, during which V, M are constant

then the amount of mass that strikes the bucket is all the mass that will catch up to it in Δt

the distance that any point on the stream travels in Δt is $V_s \Delta t$

the distance that the bucket travels in Δt is $V(t)\Delta t$

thus, the length of sand that catches up to the bucket, $\Delta l = (V_s - V(t))\Delta t$

the mass of this length of sand, $\Delta m = \frac{dm}{dl}\Delta l = \lambda(V_s - V(t))\Delta t$

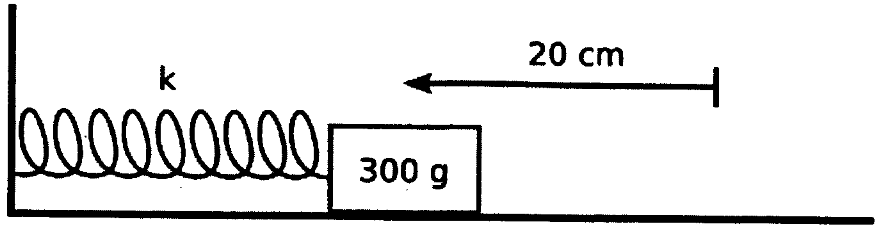
dividing by Δt gives $\frac{\Delta m}{\Delta t} = \lambda(V_s - V(t))$

if we take the limit as $\Delta t \to 0$, we have $\boxed{\frac{dm}{dt} = \lambda(V_s - V(t))}$

5. A 300-g wood block is firmly attached to a horizontal spring. The block can slide along a table with a coefficient of friction 0.40. A force of 20 N compresses the spring 20 cm. Assume $\mu_s = \mu_k$.

a) (10 points) If the spring is released from this position, how far beyond its equilibrium position will it stretch on its first swing?

b) (10 points) What total distance will the block travel and how much thermal energy will be produced by the time it comes to rest?



a) First, find the spring constant, k.

They tell you: $20N = k\,(20cm)$

$$\Rightarrow \quad k = \frac{20N}{20cm} = 100\ \frac{kg}{s^2}$$

Use the work-energy theorem to find its maximum displacement: $\Delta E = W_{Ext.}$ note: $x_i = -20$ cm

$$E_{Initial} = \frac{1}{2} k x_i^2 = \frac{1}{2}(100\ \tfrac{kg}{s^2})(-0.2\ m)^2 = 2 \text{ Joules}$$

$$E_{final} = \frac{1}{2}\underbrace{\cancel{m v_f^2}}_{0} + \frac{1}{2} k x_f^2$$

Hence: $\frac{1}{2} k x_f^2 - (2J) = -\mu_k m g\,(x_f - x_i) = -(0.4)(0.3\,kg)(10\,m/s^2)(x_f + 0.2\,m)$

$$\Rightarrow \frac{1}{2}(100\ \tfrac{kg}{s^2})x_f^2 - (2J) = -0.24\,J - (1.2\ \tfrac{kg\,m}{s^2})x_f$$

$$(50\ \tfrac{kg}{s^2})x_f^2 + (1.2\ \tfrac{kg\,m}{s^2})x_f - 1.76\,J = 0$$

$$\Rightarrow \boxed{x_f = 17.6\ cm}$$ (solve quadratic formula)

b) Since all the energy is lost to friction,

$\boxed{\text{Thermal Energy Produced} = \text{Initial Energy} = 2.0J}$

Furthermore, $E_{final} = 0 \Rightarrow \Delta E = E_f - E_i = -2.0J = -\mu_k m g X_{total}$

$$\Rightarrow \boxed{X_{total} = \frac{2.0J}{(0.4)(0.3kg)(10\,m/s^2)} = 1.67\ m}$$