Math 54
Fall 2017
Exam 2
10/31/17
Time Limit: 80 Minutes

Name: ______________________

Student ID: ______________________

GSI or Section: ______________________

This exam contains 6 pages (including this cover page) and 7 problems. Problems are printed on both sides of the pages. Enter all requested information on the top of this page.

> This is a closed book exam. No notes or calculators are permitted.
> We will drop your lowest scoring question for you.

You are required to show your work on each problem on this exam. **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

If you need more space, there are blank pages at the end of the exam. Clearly indicate when you have used these extra pages for solving a problem. However, it will be greatly appreciated by the GSIs when problems are answered in the space provided, and your final answer **must** be written in that space. Please do not tear out any pages.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

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1. (10 points) Let $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \middle| x_1 + x_2 + 2x_3 = 0 \right\}$ and $W = \text{Nul} \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 4 & 6 & 2 \end{bmatrix}$.

(a) (3 points) What is the dimension of V?

(b) (3 points) What is the dimension of W?

(c) (4 points) What is the dimension of the intersection of V and W?

2. (a) (5 points) Find all values of c such that the following matrix is diagonalizable

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & c \\ 0 & 0 & -1 \end{bmatrix}$$

(b) (5 points) For the values of c found in part (a), find an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

3. (10 points) Label the following statements as True or False. The correct answer is worth 1 point and a brief justification is worth 1 point. Credit for the justification can only be earned in conjunction with a correct answer. No points will be awarded if it is not clear whether you intended to mark the statement as True or False.

   (a) (2 points) If A and B are 2×2 matrices with $\det(A) = \det(B) = 0$, then $\{A, B\}$ is linearly dependent in the vector space of 2×2 matrices, $M_{2\times 2}$.

   (b) (2 points) If A is an $n \times n$ matrix and 0 is an eigenvalue of A, then A is not diagonalizable.

   (c) (2 points) Suppose $u, v, w \in \mathbb{R}^3$. If u is orthogonal to v and v is orthogonal to w, then u is orthogonal to w.

   (d) (2 points) If A and B are diagonalizable 2×2 matrices, then $A + B$ is diagonalizable.

   (e) (2 points) Suppose $\mathcal{B}$ and $\mathcal{C}$ are bases of $\mathbb{R}^n$. If $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ and $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the change of coordinates matrices from $\mathcal{C}$ to $\mathcal{B}$ and from $\mathcal{B}$ to $\mathcal{C}$, respectively, then $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\,\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the $n \times n$ identity matrix.

4. (a) (2 points) State the rank theorem for an $m \times n$ matrix A. (*Hint:* This theorem might be useful in both parts (b) and (c).)

(b) (4 points) Suppose that B is a 3×3 diagonalizable matrix whose characteristic polynomial is $\lambda^2(1-\lambda)$. Find the rank of B.

(c) (4 points) Suppose that C is a 4×4 matrix such that $C^2 = 0$. Show that $\text{Rank}(C) \leq 2$. (*Hint:* Show that $\text{Col}(C) \subset \text{Nul}(C)$ and look at the hint for part (a).)

5. (10 points) Consider the subspace of $\mathbb{R}^4$ below.

$$W = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

(a) (5 points) Apply the Gram-Schmidt algorithm to produce an orthogonal basis of W.

(b) (5 points) Find the distance between $v = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$ and W.

6. (a) (5 points) Show that

$$\mathcal{B} = \{1 + 5t^2, 2 + t + 6t^2, 3 + 4t\}$$

is a basis for the vector space $\mathbb{P}_2$ of polynomials of degree less than or equal to 2.

(b) (5 points) Compute the change of coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{C}$ where $\mathcal{B} = \{1 + 5t^2, 2 + t + 6t^2, 3 + 4t\}$ is as in part (a) and $\mathcal{C}$ is the following basis of $\mathbb{P}_2$.

$$\mathcal{C} = \{1 + t, t + t^2, 1\}$$

7. (a) (5 points) If A is a 3×3 matrix with real entries such that 2 and $3 - i$ are eigenvalues of A, compute $\det(A)$. (Note: It is **NOT** enough to get full credit if you only compute the determinant on a specific example that has these eigenvalues.)

(b) (5 points) Suppose that A and B are $n \times n$ matrices and B is invertible. Show that there is a $\lambda \in \mathbb{C}$ such that $A + \lambda B$ is not invertible.

Extra space.

Extra space.