EE 16B Midterm 2, March 21, 2017

Name: ________________
SID #: ________________
Discussion Section and TA: ________________
Lab Section and TA: ________________
Name of left neighbor: ________________
Name of right neighbor: ________________

Important Instructions:
• **Show your work.** An answer without explanation is not acceptable and does not guarantee any credit.
• **Only the front pages will be scanned and graded.** You can use the back pages as scratch paper.
• **Do not remove pages,** as this disrupts the scanning. Instead, cross the parts that you don’t want us to grade.

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1. (10 points) The thirteenth century Italian mathematician Fibonacci described the growth of a rabbit population by the recurrence relation:

\[ y(t+2) = y(t+1) + y(t) \]

where \( y(t) \) denotes the number of rabbits at month \( t \). A sequence generated by this relation from initial values \( y(0), y(1) \) is known as a Fibonacci sequence.

a) (5 points) Bring the recurrence relation above to the state space form using the variables \( x_1(t) = y(t) \) and \( x_2(t) = y(t+1) \).
b) (5 points) Determine the stability of this system.
2. (15 points) Consider the circuit below that consists of a capacitor, an inductor, and a third element with the nonlinear voltage-current characteristic:

\[ i = -v + v^3. \]

![Circuit Diagram]

a) (5 points) Write a state space model of the form

\[
\frac{dx_1(t)}{dt} = f_1(x_1(t), x_2(t)) \\
\frac{dx_2(t)}{dt} = f_2(x_1(t), x_2(t))
\]

using the states \( x_1(t) = v_C(t) \) and \( x_2(t) = i_L(t) \).

\[ f_1(x_1, x_2) = \quad f_2(x_1, x_2) = \]
b) (5 points) Linearize the state model at the equilibrium $x_1 = x_2 = 0$ and specify the resulting $A$ matrix.
c) (5 points) Determine stability based on the linearization.
3. (10 points) Consider the discrete-time system

\[ \ddot{x}(t + 1) = A \dot{x}(t) + Bu(t) \]

where

\[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \]

a) (5 points) Determine if the system is controllable.
b) (5 points) Explain whether or not it is possible to move the state vector from $\bar{x}(0) = 0$ to

$$\bar{x}(T) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

If your answer is yes, specify the smallest possible time $T$ and an input sequence $u(0), \ldots, u(T-1)$ to accomplish this task.
4. (20 points) Consider the system

\[ \ddot{x}(t + 1) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \]

where \( \theta \) is a constant.

a) (5 points) For which values of \( \theta \) is the system controllable?

b) (10 points) Select the coefficients \( k_1, k_2 \) of the state feedback controller

\[ u(t) = k_1 x_1(t) + k_2 x_2(t) \]

such that the closed-loop eigenvalues are \( \lambda_1 = \lambda_2 = 0 \). Your answer should be symbolic and well-defined for the values of \( \theta \) you specified in part (a).
Additional workspace for Problem 4b.
c) (5 points) Suppose the state variable $x_1(t)$ evolves as depicted below when no control is applied ($u = 0$). What is the value of $\theta$?
5. (15 points) Consider the inverted pendulum below, where \( p(t) \) is the position of the cart, \( \theta(t) \) is the angle of the pendulum, and \( u(t) \) is the input force.

When linearized about the upright position, the equations of motion are

\[
\begin{align*}
\ddot{p}(t) &= -\frac{m}{M} g \theta(t) + \frac{1}{M} u(t) \\
\ddot{\theta}(t) &= \frac{M + m}{M \ell} g \theta(t) - \frac{1}{M \ell} u(t)
\end{align*}
\] (1)

where \( M, m, \ell, g \) are positive constants.

a) (5 points) Using (1) write the state model for the vector

\[
\bar{x}(t) = [p(t) \quad \dot{p}(t) \quad \theta(t) \quad \dot{\theta}(t)]^T.
\]
b) (5 points) Suppose we measure only the position; that is, the output is $y(t) = x_1(t)$. Determine if the system is observable with this output.
c) (5 points) Suppose we measure only the angle; that is, the output is \( y(t) = x_3(t) \). Determine if the system is observable with this output.
6. (15 points) Consider the system

\[
\begin{bmatrix}
x_1(t+1) \\
x_2(t+1) \\
x_3(t+1)
\end{bmatrix} =
\begin{bmatrix}
0.9 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix}, \quad y(t) =
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix}.
\]

a) (5 points) Select values for \(\ell_1, \ell_2, \ell_3\) in the observer below such that \(\hat{x}_1(t), \hat{x}_2(t), \hat{x}_3(t)\) converge to the true state variables \(\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t)\) respectively.

\[
\begin{bmatrix}
\hat{x}_1(t+1) \\
\hat{x}_2(t+1) \\
\hat{x}_3(t+1)
\end{bmatrix} =
\begin{bmatrix}
0.9 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{x}_1(t) \\
\hat{x}_2(t) \\
\hat{x}_3(t)
\end{bmatrix} +
\begin{bmatrix}
\ell_1 \\
\ell_2 \\
\ell_3
\end{bmatrix}
\begin{bmatrix}
\hat{x}_2(t) - y(t)
\end{bmatrix}.
\]
Additional workspace for Problem 6a.
b) (5 points) Professor Arcak found a solution to part (a) that guarantees convergence of $\hat{x}_3(t)$ to $x_3(t)$ in one time step; that is
\[
\hat{x}_3(t) = x_3(t) \quad t = 1, 2, 3, \ldots
\]
for any initial $\vec{x}(0)$ and $\hat{x}(0)$. Determine his $\ell_3$ value based on this behavior of the observer. Explain your reasoning.
c) (5 points) When Professor Arcak solved part (a), he found the convergence of $\hat{x}_1(t)$ to $x_1(t)$ to be rather slow no matter what $L$ he chose. Explain the reason why no choice of $L$ can change the convergence rate of $\hat{x}_1(t)$ to $x_1(t)$.
7. (15 points) Consider a system with the symmetric form
\[
\frac{d}{dt} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} = \begin{bmatrix} F & H \\ H & F \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} G \\ G \end{bmatrix} \bar{u}(t),
\] (2)
where \( \bar{x}_1 \) and \( \bar{x}_2 \) have identical dimensions and, therefore, \( F \) and \( H \) are square matrices.

a) (5 points) Define the new variables
\[
\bar{z}_1 = \bar{x}_1 + \bar{x}_2 \quad \text{and} \quad \bar{z}_2 = \bar{x}_1 - \bar{x}_2,
\]
and write a state model with respect to these variables:
\[
\frac{d}{dt} \begin{bmatrix} \bar{z}_1(t) \\ \bar{z}_2(t) \end{bmatrix} = \begin{bmatrix} & & \\ & & \end{bmatrix} \begin{bmatrix} \bar{z}_1(t) \\ \bar{z}_2(t) \end{bmatrix} + \begin{bmatrix} \\ \end{bmatrix} \bar{u}(t).
\]
b) (5 points) Show that the system (2) is not controllable.
c) (5 points) Write a state model for the circuit below using the inductor currents as the variables. Show that the model has the symmetric form (2).