This is a closed book, closed notes exam. You need to justify every one of your answers unless you are asked not to do so. Completely correct answers given without justification will receive little credit. Look over the whole exam to find problems that you can do quickly. Hand in this exam before you leave.

80 min. 75 points in total. The raw score will be normalized according to the course policy to count into the final score.

DO NOT tear out any page or add any page. This is crucial for the grading process with gradescope. Write your name on the top left corner of each page. If your answer appears in the scratch paper appended in the end, refer to your answer using the page number.

Your name: ____

Your SID : ____

1. (15 points. This problem contains 3 questions.)
Let

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Answer the following questions

(a) Do the columns of A span $\mathbb{R}^3$? Justify your answer.

Sol: Reduce the matrix $[\vec{a}_1, \vec{a}_2, \vec{a}_3]$ to REF. One possible form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

There is a pivot in each row. So the solution of $A\vec{x} = \vec{b}$ has a solution for any $\vec{b} \in \mathbb{R}^3$, and the columns of A spans $\mathbb{R}^3$.

(b) Are the columns of A linearly independent? Justify your answer.

Sol: There is a pivot in each column of the REF, and therefore there is no free variable. So the columns of A are linearly independent.

(c) Represent $\vec{b}$ as the linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Is this representation unique? Justify your answer.

Sol: Solve the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

by reducing to REF

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}.$$

Further reduce this RREF (or use backward substitution), and the answer is

$$\vec{x} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}.$$

2. (25 points. This problem contains 5 questions.)

True or False: If True, explain why. If False, give an explicit numerical example for which the statement does not hold.

(a) $A \in \mathbb{R}^{n\times n}$ is invertible, then A^{-1} is invertible.

Sol: True. By definition $A^{-1}A = AA^{-1} = I$, hence A is the inverse matrix of A^{-1}.

(b) If n vectors in $\mathbb{R}^m$ are linearly dependent, then any vector can be represented by the linear combination of other $n-1$ vectors ($n > 1$).

Sol: False. Consider $\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,$\vec{a}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,$\vec{a}_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Then $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ are linearly dependent, but $\vec{a}_1$ cannot be represented as the linear combination of $\vec{a}_2, \vec{a}_3$.

(c) $A \in \mathbb{R}^{n\times n}$, then $(A^T)^2 = (A^2)^T$.

Sol: True. Use

$$(AB)^T = B^T A^T$$

and let $A = B$.

(d) Every subspace of $\mathbb{R}^n$ contains at most n vectors.

Sol: False. For example, $\mathbb{R}^2$ contains infinite number of vectors.

(e) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^n$. $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent, then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent.

Sol: True. If $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent, then there exists a nonzero vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so that

$$x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{0}.$$

Therefore

$$x_1\vec{v}_1 + x_2\vec{v}_2 + 0\vec{v}_3 = \vec{0}.$$

This means that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent.

3. (10 points. This problem contains 2 questions.)
a) Compute $C = A^T B$, where

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Sol:

$$C = \begin{bmatrix} 2 & 4 \\ 5 & 7 \\ 6 & 9 \end{bmatrix}$$

b) Compute the matrix inverse of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Sol: Solve the augmented problem

$$[A|I] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Reduce A to RREF and we have

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{bmatrix}.$$

Hence

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

4. (25 points. This problem contains 5 questions.)
A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ has the following effect

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Compute

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

Sol: First write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as the linear combination of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This means solving

$$\alpha_{11}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \alpha_{21}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\alpha_{12}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \alpha_{22}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This can be solved by the following augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}.$$

Reduce the left to RREF, we have

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

In other words

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Using that T is a linear transformation

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \frac{1}{2}T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) + \frac{1}{2}T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix},$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = -\frac{1}{2}T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) + \frac{1}{2}T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}.$$

(b) Write down the standard matrix of T, denoted by A.

Sol: The standard matrix is

$$A = \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(c) Find a basis for the null space and column space of A.

Sol: Reduce A to REF, and one possible form is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

There is no free column, and the null space is $\{\vec{0}\}$, and there is no basis. (if you say $\vec{0}$ is its basis, your points will not be reduced either)

Each column is a pivot column, and therefore a basis of the column space is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix} \right\}.$$

(d) Is T injective? Is T surjective? Justify your answer.

Sol: The null space of T is $\{\vec{0}\}$ and therefore T is injective.

T maps $\mathbb{R}^2$ to $\mathbb{R}^3$ and therefore cannot be surjective.

(e) State the rank theorem, and verify the rank theorem for A from the computation in (c).

Sol: The rank theorem states that the number of columns of A equals to the sum of the dimension of the null space of A and the dimension of the column space of A (i.e. the rank of A).

In this case, the number of columns is 2. The dimension of the null space of A is 0, and the dimension of the column space is 2. Hence $2 = 0 + 2$ and the rank theorem is satisfied.