University of California, Berkeley
Mechanical Engineering
ME 106, Fluid Mechanics
ODK 1, Spring 2017

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Discussion GSI: ______________

Notes:

- Your solution procedure should be legible and complete for full credit (use scratch paper as needed).
- You may use a calculator with simple arithmetic operations.

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1. A sealed water bottle of cross-sectional area $A = 4.0 \times 10^{-3}$ m^2 is filled with water. It also contains air at atmospheric pressure (1 atm) that fills a volume of $V_0 = 2 \times 10^{-4}$ m^3. The water bottle is brought to a mountain where the external air pressure is 0.90 atm. Then the bottle is punctured 10.0 cm below the water level. Assume the hole has a diameter of 1 mm. Note that 1 atm = 1.013×10^5 N/m^2.

   (a) What is the velocity of the water coming out of the hole immediately after the puncture?

   (b) Does the answer in part (a) depend on the size of the hole? Why or why not?

   (c) How different would the answer in part (a) be compared to if the the bottle was punctured at normal atmospheric pressure ($p = 1$ atm)? Express your answer in % difference.

   (d) Assume the air acts as an ideal gas and its temperature remains constant (pV = constant). Show that the air pressure inside the bottle as a function of water height h above the hole is given by $P(h) = \frac{P_0V_0}{A(h_0-h)+V_0}$, where h_0 is the initial water height.

   (e) Using your answer from part (d), what is the velocity of the water coming out of the hole as a function of height h of the water level above the hole? (Do not assume the kinetic energy of the surface is negligible.)

Solution:

(Total points: 30)

We assume the flow can be considered steady and inviscid to use Bernoulli. The bottle is also rigid enough that there is no deformation due to pressure changes.

a) The pressure inside the bottle is $P_0 = 1.013 \times 10^5$ N/m^2. The pressure outside the bottle is $P_1 = 0.90$ atm $= 0.91\times10^5$ N/m^2. Bernoulli states that $P_0+\frac{1}{2}\rho_w v_0^2+\rho_w g z_0 = P_1+\frac{1}{2}\rho_w v_1^2+\rho_w g z_1$ (3 pts). Initially $v_0 \approx 0$ m/s and $z_0 = h_0 = 10.0$ cm (so $z_1 = 0$ cm). This gives $P_0 + \rho_w g z_0 = P_1 + \frac{1}{2}\rho_w v_1^2$ (2 pts). So

$$v_1 = \left(\frac{2(P_0 - P_1)}{\rho_w} + 2gz_0\right)^{\frac{1}{2}} \tag{1}$$

$$= \left(\frac{2(1.013 \times 10^5 - 0.91 \times 10^5)N/m^2}{1000kg/m^3} + 2 \times 9.81m/s^2 \times 0.1m\right)^{1/2} \tag{2}$$

$$= 4.7m/s(2pts) \tag{3}$$

(Note that they could alternatively solved without assuming $v_0 \approx 0$.)

b) If we ignore the velocity of the top surface, then the answer doesn't depend either on the size of the top surface or the hole. This assumption only makes sense since $A_1 << A_0$. If we do not ignore the velocity of the top surface, then the size of both matter. Namely, the surface velocity is given in terms of the puncture velocity as $v_0 = \frac{A_1}{A_0}v_1$ and this can be plugged into Bernoulli, making the equation a function of A_1. However because $A_1 << A_0$, the difference in solutions is negligible.(5 pts)

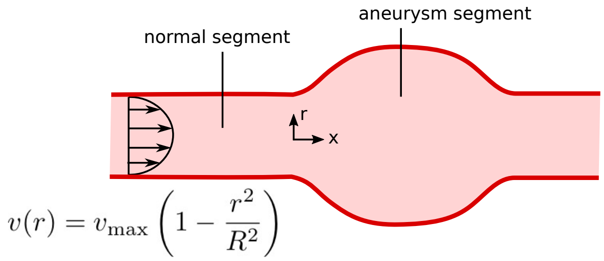
c) In this case $P_0 = P_1$ (2 pts) and $v_1 = \sqrt{2 \times 9.81 \times 0.1}$ $(2pts)$ = 1.4 m/s (1 pt). This is 336% difference. (Do not worry if they didn't express as % difference correctly.)

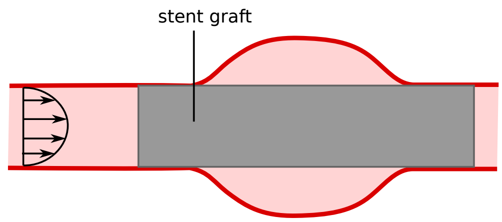
d) Let V_h (and P_h) be the air volume (and pressure) when the water level has dropped to height h. Thus $P_0V_0 = P_fV_f$. Also $V_h = A(h_0 - h) + V_0$. Then $P_h = \frac{P_0V_0}{A(h_0-h)+V_0}$. (5 pts)

e) Using the solution from the previous part, Bernoulli becomes $P_h + \frac{1}{2}\rho_w v_0^2 + \rho_w gh = P_1 + \frac{1}{2}\rho_w v_1^2$ (3 pts). Also from mass conservation $A_0v_0 = A_1v_1$, so $v_0 = \frac{A_1}{A_0}v_1$ (3 pts) where $A_1 = 2.0 \times 10{-}5$ m^2. Therefore $v_1 = ((\frac{2(P_h-P_1)}{\rho_w} + 2gh)(\frac{A_0^2}{A_0^2-A_1^2}))^{\frac{1}{2}}$ (2 pts) where P_h is as identified above. (This answer can be left in terms of known variables.)

1. (cont)

2. Consider an elastic tube in which a section of the tube becomes weaker and expands. If the tube is an artery this is known as an aneurysm. Suppose the native artery has a radius $R = 1$ cm and the blood flow has a parabolic profile given by $v(r) = v_{\max}(1 - \frac{r^2}{R^2})$ at all cross-sections, with $v_{\max} = 50$ cm/s in the normal segments.





a) Qualitatively describe how the velocity and pressure change in aneurysmal segment compared to the normal segment.

b) Suppose that the area inside of the aneurysm is approximately given by $A(x) = A_0 + \sin(\pi x)$ for $0 < x < 1$, where A_0 is area of the normal healthy portion and x is normalized axial distance along the aneurysm segment. Derive an expression for the fluid acceleration of a particle passing through the aneurysm. The particle can be considered to remain along the center of the artery ($r = 0$). (Hint: you will need to consider the continuity equation as well.)

c) As an engineer, you develop a "stent graft" to be placed inside the artery to shield the weakened section. To know if the implant will remain secured, you need to know the total shear force applied to the graft due to the blood flow. Suppose the stent graft also has a radius $R = 1$ cm and the blood flow maintains a parabolic profile inside the graft. The viscosity of blood is 0.003 Pa·s. The length of the stent graft is 10 cm.

Solution:

(Total points: 20)

a) From conservation of mass, the velocity would go down since the cross-sectional area is larger (3 pts), and thus from Bernoulli the pressure would increase since velocity decreases (3 pts). Since the pressure increased, the tube becomes more likely to fail since it has increased pressure on the weak section.

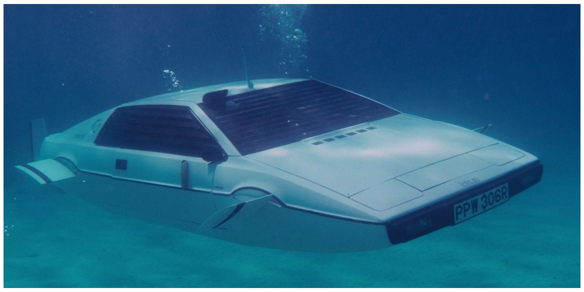
b) By continuity, the centerline velocity in the aneurysm is given by $v(x) = v_{\max}\frac{A_0}{A_0+sin(\pi x)}$ (3 pts).

The fluid acceleration is given by $a = v \cdot \frac{dv}{dx} = -v_{\max}^2 \pi \frac{A_0^2}{(A_0+\sin(\pi x))^3}\cos(\pi x)$ (3 pts).

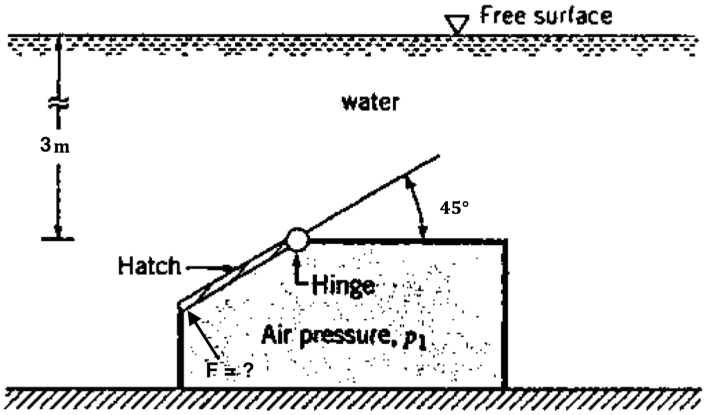
c) The wall shear stress is given as $\tau_w = -\mu \left.\frac{dv}{dr}\right|_{r=R} = \mu v_{max}\frac{2}{R}$ (3 pts) $= 0.003Ns/m^2 \times 2 \times 0.5m/s \times \frac{1}{0.01} = 0.3N/m^2$ (1 pt). The total force is $\tau_w A = \tau_w 2\pi RL$ (3 pts) $= 0.3N/m^2 \times \pi \times 0.01m \times 0.1m = 0.0019N$ or $0.01884N$ rounding to 4 sig. figs. (1 pt).

2. (cont)

3. Oh no! James Bond has crashed his submarine car. Due to a rock slide, the car is pinned on its side at a 45° angle. Unfortunately, the windows are bullet-proof and the car is perfectly sealed. Fortunately, one of the doors is unobstructed and Bond is able to increase the internal cabin pressure p_1 to 25% above atmospheric by emptying the car's oxygen tanks. Assume the door to be a 1×1 m^2 square planar surface. If the car is 3 m underwater, what is the force Bond needs to push on the end of the door with for it to begin to open? The density of water is 1000 kg/m^3. (Recall the $I_{xc} = L^4/12$ for an L $\times$ L square.)



Lotus Esprit from 10th James Bond



Engineer's rendition of Bond's crashed car

Solution: (Total Points: 20)
First calculate the centroid of area depth:

$$h_c = h_{car} + \frac{1}{2}L_{door}\sin 45^\circ (3)$$
$$h_c = 3 + \frac{1}{2}(1)\sin 45^\circ$$
$$h_c = 3.3536m (1)$$

Then calculate effective force due to the water pressure:

$$F_R = \rho g h_c A (3)$$
$$F_R = (1000)(9.81)(3.3536)(1)^2$$
$$F_R = 32.9kN (1)$$

To solve for the force Bond needs to exert, we will use the sum of moments about the door hinge. First we compute the resultant force location

$$y_R = \frac{I_{xc}}{y_c A} + y_c (2)$$

While h is vertical depth, y is the point along the line parallel to the door (at 45°). So:

$$y_c = \frac{h_{car}}{\sin 45} + \frac{1}{2}L_{door} (2)$$
$$y_c = \frac{3}{\sin 45} + 0.5$$
$$y_c = 4.7426m (1)$$

$$y_R = \frac{I_{xc}}{y_c A} + y_c$$
$$y_R = \frac{\frac{1^4}{12}}{(4.7426)(1^2)} + 4.7426$$
$$y_R = 4.7602m (1)$$

Thus, the force Bond has to exert, F_B is found using the following:
(Note that we know the internal pressure is just exerting its force at the center of the door so the moment arm is 0.5m.)

$$\sum M = 0(1)$$

$$F_R\left(y_R - \frac{3}{\sin 45}\right) - 101000 \times 0.25(1^2)0.5 - F_B(1) = 0(3)$$

$$F_B = 4402N(2)$$

3. (cont)

Summary of Equations:

Chapter 1:

Specific weight	$\gamma = \rho g$
Specific gravity	$SG = \dfrac{\rho}{\rho_{H_2O@4\,°C}}$
Ideal gas law	$\rho = \dfrac{p}{RT}$
Newtonian fluid shear stress	$\tau = \mu \dfrac{du}{dy}$
Bulk modulus	$E_v = -\dfrac{dp}{d\mathcal{V}/\mathcal{V}}$
Speed of sound in an ideal gas	$c = \sqrt{kRT}$
Capillary rise in a tube	$h = \dfrac{2\sigma \cos\theta}{\gamma R}$

Chapter 2:

Pressure gradient in a stationary fluid	$\dfrac{dp}{dz} = -\gamma$
Pressure variation in a stationary incompressible fluid	$p_1 = \gamma h + p_2$
Hydrostatic force on a plane surface	$F_R = \gamma h_c A$
Location of hydrostatic force on a plane surface	$y_R = \dfrac{I_{xc}}{y_c A} + y_c$
	$x_R = \dfrac{I_{xyc}}{y_c A} + x_c$
Buoyant force	$F_B = \gamma \mathcal{V}$
Pressure gradient in rigid-body motion	$-\dfrac{\partial p}{\partial x} = \rho a_x, \quad -\dfrac{\partial p}{\partial y} = \rho a_y, \quad -\dfrac{\partial p}{\partial z} = \gamma + \rho a_z$
Pressure gradient in rigid-body rotation	$\dfrac{\partial p}{\partial r} = \rho r \omega^2, \quad \dfrac{\partial p}{\partial \theta} = 0, \quad \dfrac{\partial p}{\partial z} = -\gamma$

Chapter 3:

Streamwise and normal acceleration	$a_s = V \dfrac{\partial V}{\partial s}, \qquad a_n = \dfrac{V^2}{\mathcal{R}}$
Force balance along a streamline for steady inviscid flow	$\displaystyle\int \frac{dp}{\rho} + \frac{1}{2}V^2 + gz = C$ (along a streamline)
The Bernoulli equation	$p + \frac{1}{2}\rho V^2 + \gamma z =$ constant along streamline
Pressure gradient normal to streamline for inviscid flow in absence of gravity	$\dfrac{\partial p}{\partial n} = -\dfrac{\rho V^2}{\mathcal{R}}$
Force balance normal to a streamline for steady, inviscid, incompressible flow	$\displaystyle p + \rho \int \frac{V^2}{\mathcal{R}}\, dn + \gamma z =$ constant across the streamline
Velocity measurement for a Pitot-static tube	$V = \sqrt{2\,(p_3 - p_4)/\rho}$
Free jet	$V = \sqrt{2\,\dfrac{\gamma h}{\rho}} = \sqrt{2gh}$
Continuity equation	$A_1 V_1 = A_2 V_2$, or $Q_1 = Q_2$
Flow meter equation	$Q = A_2 \sqrt{\dfrac{2(p_1 - p_2)}{\rho[1 - (A_2/A_1)^2]}}$
Sluice gate equation	$Q = z_2 b \sqrt{\dfrac{2g(z_1 - z_2)}{1 - (z_2/z_1)^2}}$
Total head	$\dfrac{p}{\gamma} + \dfrac{V^2}{2g} + z =$ constant on a streamline $= H$

Chapter 4:

Equation for streamlines	$\frac{dy}{dx} = \frac{v}{u}$
Acceleration	$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$
Material derivative	$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + (\mathbf{V}\cdot\nabla)(\ )$
Streamwise and normal components of acceleration	$a_s = V\frac{\partial V}{\partial s}, \qquad a_n = \frac{V^2}{\mathcal{R}}$
Reynolds transport theorem (restricted form)	$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial B_{\text{cv}}}{\partial t} + \rho_2 A_2 V_2 b_2 - \rho_1 A_1 V_1 b_1$
Reynolds transport theorem (general form)	$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{\text{cv}} \rho b\, d\mathcal{V} + \int_{\text{cs}} \rho b\, \mathbf{V}\cdot\hat{\mathbf{n}}\, dA$
Relative and absolute velocities	$\mathbf{V} = \mathbf{W} + \mathbf{V}_{\text{cv}}$