Math 1B. Solutions to the Final Exam

1. (16 points) Suppose that $|f''(x)| \le K$ for all $a \le x \le b$. If E_T is the error involved in using the Trapezoidal Rule for computing $\int_a^b f(x)\,dx$ with n subintervals, then

$$|E_T| \le \frac{K(b-a)^3}{12n^2} .$$

Using this bound, what is the smallest value of n that will guarantee that the Trapezoidal approximation for $\displaystyle\int_1^3 x^3\,dx$ is accurate to within $1/6$? (Your answer should be an integer.)

Since $f(x) = x^3$, we have $f''(x) = 6x$. On the interval $[1,3]$, the maximum of $|f''(x)| = |6x|$ occurs at $x = 3$, so we can take $K = 18$. We also have $a = 1$ and $b = 3$, so the error bound is

$$|E_T| \le \frac{18\cdot 2^3}{12n^2} = \frac{3\cdot 2^3}{2n^2} = \frac{3\cdot 2^2}{n^2} = \frac{12}{n^2} .$$

This needs to be smaller than $1/6$, so:

$$\begin{aligned} \frac{12}{n^2} &\le \frac{1}{6} ; \\ 6\cdot 12 &\le n^2 ; \\ 72 &\le n^2 . \end{aligned}$$

If $n = 8$ then $n^2 = 64$, which is not big enough, but if $n = 9$ then $n^2 = 81$, which is big enough.

Therefore the smallest possible value of n is $n = 9$.

2. (16 points) Find the area of the surface generated by rotating the curve

$$y = 5 - \frac{x^2}{2} , \qquad 1 \le x \le 2 ,$$

about the y-axis.

Let $f(x) = 5 - \frac{x^2}{2}$. Then $f'(x) = -x$, and therefore $ds = \sqrt{1+x^2}\,dx$. Therefore the area of the surface is

$$\begin{aligned} \int_a^b 2\pi x\,ds &= 2\pi \int_1^2 x\sqrt{1+x^2}\,dx \\ &= \pi \int_2^5 \sqrt{u}\,du \\ &= \frac{2}{3}\pi u^{3/2}\Big|_2^5 \\ &= \frac{2\pi}{3}\big(5\sqrt{5} - 2\sqrt{2}\big) . \end{aligned}$$

(Here we used a substitution $u = 1 + x^2$, $du = 2x\,dx$.)

3. (20 points) Determine whether the series

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))}$$

is absolutely convergent, conditionally convergent, or divergent.

Since $e < 3$, we have $\ln(\ln x) > \ln(\ln e) = \ln 1 = 0$, so the functions $\ln x$ and $\ln(\ln x)$ are positive and increasing on the interval $[3, \infty)$. In particular, the denominator of $f(x) = \frac{1}{x(\ln x)(\ln(\ln x))}$ is positive and increasing on $[3, \infty)$, and so $f(x)$ is positive and decreasing on that interval.

Therefore, we can apply the Integral Test on that interval.

Since the Integral Test involves a substitution in an improper integral, it is best to do the indefinite integral first. Using the substitutions $u = \ln x$, $du = dx/x$ and $v = \ln u$, $dv = du/u$, we have

$$\int \frac{dx}{x(\ln x)(\ln(\ln x))} = \int \frac{du}{u(\ln u)} = \int \frac{dv}{v} = \ln |v| + C = \ln(\ln(\ln x)) + C \,.$$

Therefore

$$\begin{aligned}
\int_3^{\infty} \frac{dx}{x(\ln x)(\ln(\ln x))} &= \lim_{t\to\infty} \int_3^t \frac{dx}{x(\ln x)(\ln(\ln x))} \\
&= \lim_{t\to\infty} \ln(\ln(\ln x)) \Big|_3^t \\
&= \lim_{t\to\infty} \Big(\ln(\ln(\ln t)) - \ln(\ln(\ln 3)) \Big) \\
&= \infty \,.
\end{aligned}$$

Since this improper integral diverges, the original series also diverges, by the Integral Test.

4. (18 points) Determine whether the series

$$\sum_{n=1}^{\infty} \left(\left(1 + \frac{1}{n}\right)^{\pi} - 1 - \frac{\pi}{n} \right) \sqrt{n}$$

is absolutely convergent, conditionally convergent, or divergent.

By the binomial series,

$$\left(1 + \frac{1}{n}\right)^{\pi} = 1 + \pi \left(\frac{1}{n}\right) + \binom{\pi}{2} \left(\frac{1}{n}\right)^2 + \binom{\pi}{3} \left(\frac{1}{n}\right)^3 + \dots \,;$$

therefore the n^{th} term in the series in the problem (call it a_n) is

$$a_n = \sum_{m=2}^{\infty} \binom{\pi}{m} \left(\frac{1}{n}\right)^{m-1/2} .$$

Since the dominant term (for n large) in the above series is the first term, this says that a_n behaves like $\binom{\pi}{2}\left(\frac{1}{n}\right)^{3/2}$, so it makes sense to try using the Limit Comparison Test with the convergent p-series $\sum \frac{1}{n^{3/2}}$. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{n^{3/2}} &= \lim_{n\to\infty} \left(\sum_{m=2}^{\infty} \binom{\pi}{m} \left(\frac{1}{n}\right)^{m-2} \right) \\
&= \lim_{x\to 0} \left(\sum_{m=2}^{\infty} \binom{\pi}{m} x^{m-2} \right) \\
&= \binom{\pi}{2} \\
&= \frac{\pi(\pi-1)}{2} \ne 0 ,
\end{aligned}$$

where we let $x = 1/n$. Since the series $\sum 1/n^{3/2}$ converges, so does the series given in the problem. Also, the series converges absolutely, because (for large enough n), all of the terms are positive.

5. (22 points) Find the Maclauren series for $\ln\left(\frac{1+x}{1-x}\right)$, and determine its radius of convergence and interval of convergence.

[**Hint:** Use a property of logarithms.]

Using a property of logarithms, and the given series for $\ln(1+x)$,

$$\begin{aligned}
\ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + \sum_{n=1}^{\infty} \frac{x^n}{n} .
\end{aligned}$$

When n is even, the terms cancel. This leaves only the odd terms, so we have

$$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} . \qquad (*)$$

The radius of convergence for the series for $\ln(1+x)$ is $R = 1$, so it converges for all $x \in (-1,1)$. If we replace x with $-x$ (as was done to get the series for $\ln(1-x)$), the resulting series still converges for all $x \in (-1,1)$ (because if $x \in (-1,1)$, then $-x \in (-1,1)$, too).

Therefore the series (*) converges for all $x \in (-1,1)$, and its radius of convergence is at least 1.

On the other hand, when $x = 1$, the series (*) simplifies to

$$\sum_{n=1}^{\infty} \frac{2}{2k+1} .$$

This series diverges, by using the limit comparison test with the harmonic series $\sum(1/k)$. This implies that the radius of convergence is at most 1, so $R = 1$ for the series (*). Also, $x = 1$ is not in the interval of convergence. When $x = -1$, we get minus the series for $x = 1$, so it also diverges.

Therefore the interval of convergence is $(-1,1)$.

6. (18 points) (a). Use Euler's method with step size 0.5 to estimate $y(1.5)$, where $y(x)$ is the solution of the initial-value problem

$$y' = \tan^{-1}(2x+y) , \qquad y(0) = 0 .$$

The computation is as follows:

n	x_n	y_n
0	0	0
1	0.5	$0 + 0.5\tan^{-1}(2\cdot 0 + 0) = 0$
2	1.0	$0 + 0.5\tan^{-1}(2\cdot 0.5 + 0) = \frac{1}{2}\tan^{-1} 1 = \frac{\pi}{8}$
3	1.5	$\frac{\pi}{8} + 0.5\tan^{-1}\left(2\cdot 1 + \frac{\pi}{8}\right)$

So the estimate for $y(1.5)$ is

$$y(1.5) \approx \frac{\pi}{8} + \frac{1}{2}\tan^{-1}\left(2 + \frac{\pi}{8}\right) .$$

(b). Is your estimate for $y(0.5)$ larger or smaller than the actual value of $y(0.5)$? Explain.

[**Hint:** Is the graph of $y(x)$ concave up or down? You may assume that $y(x) > -2x$ for all $x > 0$.]

To determine concavity, we look at $y''(x)$ in the interval $0 \le x \le 0.5$. To find y'', differentiate both sides of the differential equation:

$$\begin{aligned} y'' &= \frac{d}{dx}\tan^{-1}(2x+y) \\ &= \frac{2+y'}{(2x+y)^2+1} \\ &= \frac{2+\tan^{-1}(2x+y)}{(2x+y)^2+1} . \end{aligned}$$

The denominator is always positive, and the numerator is likewise positive because the given fact that $y > -2x$ implies that $2x + y > 0$, so $\tan^{-1}(2x + y) > 0$.

Therefore the graph of $y(x)$ is concave upward. Since Euler's method uses the tangent line to the graph, the actual value of $y(0.5)$ will be larger than the estimate using Euler's method.

Alternative method (unanticipated): The estimated value for $y(0.5)$ is 0, so we only need to know that $y(0.5) > 0$. But, from the given inequality $y > -2x$, we have $2x + y > 0$ for all x in $(0, 0.5)$, so $y' = \tan^{-1}(2x + y)$ is positive on this interval. Therefore $y(x)$ is increasing on this interval. Since $y(0) = 0$, this implies that $y(0.5) > 0$, so our estimate is smaller than the actual value.

7. (22 points) Find all solutions of the differential equation

$$y' = \sin x \cos^2 y \ .$$

Express your solutions in the form $y = \dots$ if possible.

This is a separable differential equation, so we solve it by separating variables and integrating:

$$\begin{aligned}
\frac{dy}{dx} &= \sin x \cos^2 y \\
\frac{dy}{\cos^2 y} &= \sin x \, dx \qquad (\cos y \neq 0) \\
\int \sec^2 y \, dy &= \int \sin x \, dx \\
\tan y &= -\cos x + C \\
y &= \tan^{-1}(C - \cos x) + n\pi \ , \quad n \in \mathbb{Z} \ .
\end{aligned}$$

If $\cos y = 0$, then $y = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$, so these are equilibrium (constant) solutions.

All together, the solutions are:

$$\begin{aligned}
& y = \tan^{-1}(C - \cos x) + n\pi \ , \quad n \in \mathbb{Z} \\
\text{and} \quad & y = \frac{\pi}{2} + n\pi \ , \quad n \in \mathbb{Z} \ .
\end{aligned}$$

8. (20 points) Solve the initial-value problem

$$y' + \frac{y}{x} = 2\,, \qquad y(6) = 2\,.$$

This is a linear equation. The integrating factor is

$$I(x) = e^{\int dx/x} = e^{\ln x} = x\,;$$

multiplying both sides of the differential equation by this factor gives

$$\begin{aligned} xy' + y &= 2x\,; \\ (xy)' &= 2x\,; \\ xy &= x^2 + C\,; \\ y &= x + \frac{C}{x}\,. \end{aligned}$$

This is the general solution. To solve the initial-value problem, plug in $x = 6$ and $y = 2$ to the equation $xy = x^2 + C$ to get $12 = 36 + C$, so $C = -24$ and the solution is

$$y = x - \frac{24}{x}\,.$$

9. (20 points) For each of the following differential equations, write a trial solution for finding y_p using the Method of Undetermined Coefficients.

(Do not solve for the coefficients.)

(a). $y'' - 3y' + 2y = xe^x + e^x \cos x$

The auxiliary equation is $r^2 - 3r + 2 = 0$, which has roots $r = 1$ and $r = 2$. Therefore the general solution to the complementary equation is

$$y_c = c_1 e^x + c_2 e^{2x}\,,$$

and the form of the trial solution is

$$y_p = (Ax^2 + Bx)e^x + e^x(C\cos x + D\sin x)\,.$$

(b). $y'' - 2y' + y = xe^x$

Here the auxiliary equation is $r^2 - 2r + 1 = 0$, which has the double root $r = 1$. Therefore $y_c = (c_1 x + c_2)e^x$, and the form of y_p is

$$y_p = (Ax^3 + Bx^2)e^x\,.$$

(c). $y'' - 2y' + 2y = e^x + \cos x$

Here the auxiliary equation is $r^2 - 2r + 2 = 0$. Solving it (using the quadratic equation) gives $r = 1 \pm i$, so $y_c = e^x(c_1\cos x + c_2\sin x)$, and the form of the trial solution for y_p is

$$y_p = Ae^x + B\cos x + C\sin x\,.$$

10. (25 points) Find the general solution of the differential equation

$$y'' + y = \tan x \sec x\ .$$

The auxiliary equation is $r^2 + 1 = 0$, which has roots $r = \pm i$. Therefore the general solution of the complementary equation is

$$y_c = c_1 \cos x + c_2 \sin x\ .$$

The right-hand side of the differential equation is not of the form for which the Method of Undetermined Coefficients can be used, so we need to use the Method of Variation of Parameters. This means looking for a solution of the form

$$y_p = u_1 \cos x + u_2 \sin x\ .$$

The equations to be solved are

$$\begin{aligned} (\cos x)u_1' + (\sin x)u_2' &= 0\ , \\ (-\sin x)u_1' + (\cos x)u_2' &= \tan x \sec x\ . \end{aligned}$$

Multiplying the first equation by $\sin x$ and the second by $\cos x$ and adding gives

$$\begin{aligned} (\sin^2 x + \cos^2 x)u_2' &= \tan x \sec x \cos x\ ; \\ u_2' &= \tan x\ . \end{aligned}$$

From the first equation, we then have

$$u_1' = -\frac{\sin x}{\cos x} u_2' = -\tan^2 x\ .$$

Therefore

$$u_1 = \int (1 - \sec^2 x)\, dx = x - \tan x \qquad \text{and} \qquad u_2 = \int \tan x\, dx = \ln|\sec x|$$

(we're just looking for a particular solution, so a constant of integration is not needed).

The general solution is then

$$\begin{aligned} y &= (x - \tan x)\cos x + (\ln|\sec x|)\sin x + c_1 \cos x + c_2' \sin x \\ &= x \cos x + (\sin x)\ln|\sec x| + c_1 \cos x + c_2 \sin x \end{aligned}$$

(where $c_2' - 1 = c_2$).

11. (28 points) Use power series to solve the initial-value problem

$$y'' - xy' - y = 0\ , \qquad y(0) = 1\ , \quad y'(0) = 0\ .$$

(You do not need to check that your answer converges.)

Let $y = \sum_{n=0}^{\infty} c_n x^n$; then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, and we have

$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \qquad \text{and} \qquad -xy' = -\sum_{n=1}^{\infty} n c_n x^n\ .$$

We can start the sum for xy' at $n = 0$ instead of $n = 1$ without throwing in any terms since the $n = 0$ term is zero. Also, we can shift the indices for the series for y'' by replacing n with $n + 2$. This gives

$$y'' = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n\ ;$$

$$-xy' = -\sum_{n=0}^{\infty} n c_n x^n\ ;$$

$$-y = -\sum_{n=0}^{\infty} c_n x^n\ .$$

Adding these series then gives

$$\sum_{n=0}^{\infty}((n+2)(n+1)c_{n+2} - nc_n - c_n)x^n = 0\ .$$

Setting each coefficient equal to zero then gives

$$(n+2)(n+1)c_{n+2} - (n+1)c_n = 0\ ;$$

$$c_{n+2} = \frac{c_n}{n+2}\ .$$

This is our recursion relation, valid for all $n \geq 0$. (Note that dividing by $n + 1$ never resulted in dividing by zero.)

The first several coefficients are therefore:

from the initial condition $y(0) = 1$ $\quad c_0 = 1$

from the initial condition $y'(0) = 0$: $\quad c_1 = 0$

from the recursion relation with $n = 0$: $\quad c_2 = \dfrac{c_0}{2} = \dfrac{1}{2}$

setting $n = 1$: $\quad c_3 = \dfrac{c_1}{3} = 0$

setting $n = 2$: $\quad c_4 = \dfrac{c_2}{4} = \dfrac{1/2}{4} = \dfrac{1}{2 \cdot 4}$

setting $n = 3$: $\quad c_5 = \dfrac{c_3}{5} = 0$

setting $n = 4$: $\quad c_6 = \dfrac{c_4}{6} = \dfrac{1}{2 \cdot 4 \cdot 6}$

The pattern is now apparent: $c_n = 0$ if n is odd, and

$$c_n = \frac{1}{2 \cdot 4 \cdot \ldots \cdot n} = \frac{1}{2^k (k!)}$$

if $n = 2k$ is even.

Therefore the solution is

$$y = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k (k!)} \ .$$