FINAL EXAM: MATH 104 SECTION 004

2016. 12. 12.
7:00 PM-9:00 PM

NOTE: (1) You may use all the basic arithmetic manipulations without proof and without mention. You may also use any result proved in class or assignment, in the official textbook, but these must be mentioned.
(2) In this exam, $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{R}$ denotes the set of real numbers.

Student Name: (First) ____________________ (Last) ____________________

Signature: ____________________

Berkeley e-mail address : ____________________ @ berkeley.edu

1. True of False? (No proofs are required.)

(1) (7 points) (TRUE/FALSE) Consider a continuous function f on $(0, \infty)$, and suppose that f is a uniformly continuous on $(0, a)$ for all $a > 0$. Then f must be a uniformly continuous function on $(0, \infty)$.

Answer:

(2) (7 points) (TRUE/FALSE) Suppose that a sequence $(a_n)_{n=1}^{\infty}$ of real numbers satisfies $\lim |a_n|^{1/n} = 1$. Then, the infinite series $\sum_{n=1}^{\infty} a_n x^n$ must converge pointwise on $(-1, 1)$, but cannot converge uniformly on $(-1, 1)$.

Answer:

(3) (7 points) (TRUE/FALSE) Consider a sequence $(f_n)_{n=1}^{\infty}$ of continuous functions on $[0, 1]$. Suppose that (f_n) converges pointwise to a function f on $[0, 1]$, and that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx.$$

Then, (f_n) must converge to f uniformly on $[0, 1]$.

Answer:

(4) (7 points) (TRUE/FALSE) Suppose that a sequence of functions $(f_n)_{n=1}^{\infty}$ converges to f uniformly on $(0, 1)$. Then, the sequence $(f_n^3)_{n=1}^{\infty}$ converges to f^3 uniformly on $(0, 1)$.

Answer:

(5) (7 points) (TRUE/FALSE) Consider a function f on a non-empty subset S of $\mathbb{R}$. For $a \in S$, suppose that the function f satisfies $\lim_{x \to a^S} f(x) = L$. Let T be a non-empty subset of S containing a. Then, $\lim_{x \to a^T} f(x) = L$.

Answer:

(6) (7 points) (TRUE/FALSE) Let $0 < a < 1$ be a fixed number. Suppose that a sequence of functions $(g_n)_{n=1}^{\infty}$ on $[0,\, a]$ satisfies $|g_n(x)| \leq x^n$ for all $x \in [0,\, a]$ and for all $n \in \mathbb{N}$. Then, $\sum g_n(x)$ is a uniformly convergent infinite series.

Answer:

(7) (7 points) (TRUE/FALSE) Suppose that a set S with a distance function d is a complete metric space. Let $(s_n)_{n=1}^{\infty}$ be a Cauchy sequence in this metric space, and let $f : S \to \mathbb{R}$ be a continuous function. Then $(f(s_n))_{n=1}^{\infty}$ must be a Cauchy sequence of real numbers.

Answer:

(8) (7 points) (TRUE/FALSE) For $x = (x_1,\, x_2) \in \mathbb{R}^2$ and $y = (y_1,\, y_2) \in \mathbb{R}^2$, let us define $d(x,\, y) = |x_1 - y_1|$. Then, d is a distance function on $\mathbb{R}^2$.

Answer:

2. Consider the following power series. (No proofs are required.)

$$\sum_{n=0}^{\infty} \frac{3^n}{2^n + n} x^n$$

(1) (8 points) Find the radius of convergence.

Answer:

(2) (8 points) Determine the exact interval of convergence.

Answer:

3. Find the following limits (No proofs are required.)

(1) (8 points)

$$\lim_{x \to 0^-} \frac{x^3 + x}{|x|}$$

Answer:

(2) (8 points)

$$\lim_{x \to 1} \frac{(x-1)^2}{x - \sqrt{2x-1}}$$

Answer:

4. (10 points) State the definition of the following notion:

Let S be a subset of $\mathbb{R}$, let a be a real number that is the limit of some sequence in S, and let L be a real number. We write $\lim_{x \to a^S} f(x) = L$ if:

5. (10 points) State the Weierstrass M-test.

6. Determine whether the following functions are uniformly continuous or not, and prove your answer.

(1) (10 points) The function $f(x) = \sqrt{x}$ on (0, 1).

(2) (10 points) The function $f(x) = \frac{x^2}{1+x}$ on $(1, \infty)$.

7. Define a sequence of functions $(f_n)_{n=1}^{\infty}$ on $[0, 2]$ by

$$f_n(x) = \frac{\sqrt{x^2 + n}}{\sqrt{x + n}} \, .$$

(1) (12 points) Prove that the sequence $(f_n)_{n=1}^{\infty}$ converges to some function f uniformly on $[0, 2]$.

(2) (5 points) Find $\lim_{n\to\infty} \int_0^2 f_n(x)dx$ and justify your answer.

8. Define a sequence of functions $(g_n)_{n=1}^{\infty}$ by

$$g_n(x) = \frac{x^{n+1} + 2}{(n^2 + \cos n)(x^n + 1)}.$$

(1) (12 points) For $a > 0$, prove that $\sum_{k=1}^{\infty} g_k(x)$ converges to a continuous function on $[0, a]$.

(2) (5 points) Prove that $\sum_{k=1}^{\infty} g_k(x)$ converges to a continuous function on $[0, \infty)$.

9. (14 points) Suppose that a sequence of functions $(f_n)_{n=1}^{\infty}$ on $[0, 1]$ converges uniformly to f on $[0, 1]$. Let g be a continuous function on $[0, 1]$. Prove that $(f_n g)_{n=1}^{\infty}$ converges uniformly to fg on $[0, 1]$.

10. Suppose that we have two different distance functions d_1 and d_2 for a set S. The metric space S with distance function d_1 is denoted by (S, d_1), and the metric space S with distance function d_2 is denoted by (S, d_2). Let us assume that the metric space (S, d_1) is complete, and that

$$d_1(x, y) \leq d_2(x, y) \leq 2d_1(x, y)$$

for all $x, y \in S$.

(1) (8 points) Suppose that $(s_n)_{n=1}^{\infty}$ is a Cauchy sequence in the metric space (S, d_2). Prove that $(s_n)_{n=1}^{\infty}$ is a Cauchy sequence in the metric space (S, d_1) as well.

(2) (8 points) Suppose that the sequence $(s_n)_{n=1}^{\infty}$ converges to $s \in S$ in the metric space (S, d_1). Prove that $(s_n)_{n=1}^{\infty}$ converges to s in the metric space (S, d_2) as well.

(3) (6 points) Prove that the metric space (S, d_2) is complete.