MATH 185-4 FINAL SOLUTION

1. (8 points) Determine whether the following statements are true of false, no justification is required.

(1) (1 point) Let D be a domain and let $u, v : D \to \mathbb{R}$ be two harmonic functions, then $u + iv$ is analytic on D.

False. u and v should satisfy the Cauchy–Riemann equation. For example, for harmonic function $u(x, y) = x$ and $v(x, y) = 0$, $u + iv$ is not analytic.

(2) (1 point) Let D be a domain and let $f : D \to \mathbb{C}$ be an analytic function, then $f(z)\mathrm{d}z$ is a closed form on D.

True. This is a theorem on the book. You can also directly check that $f(z)\mathrm{d}z = f(x, y)\mathrm{d}x + if(x, y)\mathrm{d}y$ is a closed form, by using Cauchy–Riemann equation.

(3) (1 point) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. Then for any $z_0 \in \mathbb{C}$, $f(z)$ equals a power series centered at z_0, and the radius of convergence of this power series is ∞.

True. This is a theorem on the book: analytic functions have power series expansions, and the radius of convergence is the radius of the disc where f is (can be extended to) analytic.

(4) (1 point) If $f : \{1 < |z| < 2\} \to \mathbb{C}$ is an analytic function, then there exist analytic functions $f_1 : \{|z| < 2\} \to \mathbb{C}$ and $f_2 : \{|z| > 1\} \to \mathbb{C}$ such that $f = f_1 - f_2$ on $\{1 < |z| < 2\}$.

True. This is just the Laurant decomposition.

(5) (1 point) For the function $f(z) = \sin \frac{1}{z}$, $z = 0$ is a pole.

False. If $z = 0$ is a pole, $\lim_{z\to 0} \sin \frac{1}{z}$ should be ∞. However, for $\{\frac{1}{n\pi}\}$, we have $\frac{1}{n\pi} \to 0$ and $f(\frac{1}{n\pi}) = \sin n\pi = 0$. Moreover, this is actually an essential singularity, since for $\{\frac{1}{(2n+\frac{1}{2})\pi}\}$, we have $f(\frac{1}{(2n+\frac{1}{2})\pi}) = 1$.

(6) (1 point) Let D be a domain and let $f : D \to \mathbb{C}$ be an analytic function, then for any $z_0 \in D$, the residue satisfies $\text{Res}[f(z), z_0] = 0$.

True. The residue at z_0 is the coefficient of the $(z - z_0)^{-1}$ term in the Laurent expansion. Since f is analytic, there is always no $(z - z_0)^{-1}$ term.

(7) (1 point) Let D be a bounded domain with piecewise smooth boundary, and let f be a meromorphic function on D that extends to be analytic on ∂D and never vanishes on ∂D. If $\oint_{\partial D} \frac{f'(z)}{f(z)} \mathrm{d}z = 4\pi i$, then f has exactly two zeros in D.

False. The statement only holds for analytic functions, but not for meromorphic functions. For a meromorphic function, the line integral equals $4\pi i$ implies that the number of zeros minus the number of poles is equal to 2.

(8) (1 point) Let $\{f_n\}$ be a sequence of univalent analytic functions defined on a domain D, and $\{f_n\}$ uniformly converges to $f : D \to \mathbb{C}$, then f is also univalent.

False. It is possible that f is a constant function. For example, we can take $D = \{|z| < 1\}$ and $f_n(z) = \frac{z}{n}$, then the uniform limit is $f(z) = 0$.

2. (10 points) Please briefly answer the following questions.

(1) (2 points) Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ be a continuous function. What is the definition of "f is an analytic function on D"?

For any point $z_0 \in D$, the complex derivative $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exits, and the complex derivative $f' : D \to \mathbb{C}$ is a continuous function.

(2) (2 points) Let $f : \{|z - z_0| < 2\} \to \mathbb{C}$ be an analytic function, please write $f(z_0)$ as a complex line integral. (Cauchy's integral formula)

Cauchy integral formula:

$$f(z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{z - z_0} \mathrm{d}z, \text{ any } r \in (0, 2).$$

(3) (2 points) Let $f : \{1 < |z| < 2\} \to \mathbb{C}$ be an analytic function, then what is the Laurent expansion of f?

The Laurent expansion is to express f as $\sum_{k=-\infty}^{\infty} a_k z^k$, where

$$a_k = \frac{1}{2\pi i} \oint_{|w|=r} f(w) w^{-k-1} \mathrm{d}w$$

for any $r \in (1, 2)$.

(4) (2 points) Let $f : \{0 < |z - z_0| < 1\} \to \mathbb{C}$ be an analytic function, then what is the residue of f at z_0? Please also write it as a complex line integral. ba
The residue of f at z_0 is the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion

of f, which is equal to $\frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z) \mathrm{d}z$ for any $r \in (0, 1)$.

(5) (2 points) Let D be a bounded domain with piecewise smooth boundary, and let $f : D \to \mathbb{C}$ be an analytic function that extends to be analytic on ∂D and never equals 0 on ∂D. Please write the number of zeros of f in D as a complex line integral.

The number of zeros of f in D is equal to

$$\frac{1}{2\pi i} \oint_{\partial D} \frac{f'(z)}{f(z)} \mathrm{d}z.$$

3. (10 points) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function, and suppose

$$f(z) = u(x, y) + iv(x, y)$$

for two real valued functions u and v. Please show that

$$g(z) = u(x, -y) - iv(x, -y)$$

is also an analytic function on $\mathbb{C}$.

Let $U(x, y) = u(x, -y)$ and $V(x, y) = -v(x, -y)$. Then we need to check that $g(z) = U(x, y) + iV(x, y)$ satisfies the Cauchy–Riemann equation.

Since f is analytic, we know that the Cauchy–Riemann equation holds:

$$u_x(x, y) = v_y(x, y) \text{ and } u_y(x, y) = -v_x(x, y).$$

For the function $g = U + iV$, we have $U_x(x, y) = u_x(x, -y)$ and $V_y(x, y) = v_y(x, -y) = u_x(x, -y)$, so $U_x(x, y) = V_y(x, y)$ holds. We also have $U_y(x, y) = -u_y(x, -y)$ and $V_x(x, y) = -v_x(x, -y) = u_y(x, -y)$, so $U_y(x, y) = -V_x(x, y)$ holds. This shows that U and V satisfy the Cauchy–Riemann equation.

Since f is analytic on $\mathbb{C}$, all the first order partial derivatives of u and v are continuous. The above equalities imply that all first order partial derivatives of U and V are also continuous. So g is an analytic function.

Alternatively, you can check that $g(z) = \overline{f(\overline{z})}$ and it is an analytic function. For the complex derivative

$$g'(z) = \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{f(\overline{z + \Delta z}) - f(\overline{z})}}{\Delta z} = \lim_{\Delta z \to 0} \overline{\left(\frac{f(\overline{z} + \overline{\Delta z}) - f(\overline{z})}{\overline{\Delta z}} \right)} = \overline{f'(\overline{z})}.$$

So the complex derivative of $g(z)$ exists, which is equal to $\overline{f'(\overline{z})}$. This is a continuous function since f' is continuous.

4. (10 points) Please compute the complex line integral

$$\oint_{|z|=3} \frac{\cos z}{(z-1)(z+1)^2} \mathrm{d}z.$$

The zeros of the denominator are $z = 1$ and $z = -1$. So for any small $\epsilon > 0$ (say smaller than 1), since $\frac{\cos z}{(z-1)(z+1)^2}$ is analytic on $\{|z| < 3\} \setminus \{\pm 1\}$, we have

$$\oint_{|z|=3} \frac{\cos z}{(z-1)(z+1)^2} \mathrm{d}z = \oint_{|z-1|=\epsilon} \frac{\cos z}{(z-1)(z+1)^2} \mathrm{d}z + \oint_{|z+1|=\epsilon} \frac{\cos z}{(z-1)(z+1)^2} \mathrm{d}z.$$

For the first term, we have

$$\frac{1}{2\pi i} \oint_{|z-1|=\epsilon} \frac{\cos z}{(z-1)(z+1)^2} \mathrm{d}z = \frac{1}{2\pi i} \oint_{|z-1|=\epsilon} \frac{\frac{\cos z}{(z+1)^2}}{(z-1)} \mathrm{d}z = \Big(\frac{\cos z}{(z+1)^2}\Big)\Big|_{z=1} = \frac{\cos 1}{4}.$$

For the second term, we have

$$\frac{1}{2\pi i} \oint_{|z+1|=\epsilon} \frac{\cos z}{(z-1)(z+1)^2} \mathrm{d}z = \frac{1}{2\pi i} \oint_{|z+1|=\epsilon} \frac{\frac{\cos z}{z-1}}{(z+1)^2} \mathrm{d}z = \Big(\frac{\cos z}{z-1}\Big)'\Big|_{z=-1} = -\frac{\sin 1}{2} - \frac{\cos 1}{4}.$$

So

$$\oint_{|z|=3} \frac{\cos z}{(z-1)(z+1)^2} \mathrm{d}z = 2\pi i (\frac{\cos 1}{4} - \frac{\sin 1}{2} - \frac{\cos 1}{4}) = -i\pi \sin 1.$$

5. (10 points) Let D be a domain, and let $f, g : D \to \mathbb{C}$ be two analytic functions. Suppose that g is not constantly zero and $|f(z)| \le |g(z)|$ for all $z \in \mathbb{D}$.

(1) (2 points) On which subset of D is the function $h(z) = \frac{f(z)}{g(z)}$ defined? (It is possible that $g(z) = 0$ for some $z \in D$.)

The function $h(z) = \frac{f(z)}{g(z)}$ is defined on the subset of D where g is not equal to 0, i.e.

$$\{z \in D \mid g(z) \neq 0\}.$$

(2) (8 points) Please show that $h(z)$ can be extended to an analytic function on D.

Since $\{z \in D \mid g(z) \neq 0\}$ is an open set, and both f and g are analytic, $h(z) = \frac{f(z)}{g(z)}$ is analytic on $\{z \in D \mid g(z) \neq 0\}$. So we need only to take care of the zero set of g, i.e. $Z_g = \{z \in D \mid g(z) = 0\}$.

Since g is not constantly zero, the zero set Z_g consists of isolated points. So for any $z_0 \in Z_g$, there exists $\epsilon > 0$ such that $\{|z - z_0| < \epsilon\} \cap Z_g = \{z_0\}$.

On the set $\{0 < |z - z_0| < \epsilon\}$, $h(z) = \frac{f(z)}{g(z)}$ is an analytic function, and $|h(z)| = |\frac{f(z)}{g(z)}| \le 1$. Since h is bounded near z_0, z_0 is a removable singularity of $h(z)$. So we can extend $h(z)$ to z_0 analytically.

This shows that all the zeros of $g(z)$ in D are removable singularities of $h(z)$. So we can extend h to an analytic function on D.

6. (10 points) Please compute

$$\int_0^\infty \frac{\sqrt{x}}{x^4+16}\mathrm{d}x.$$

(Your answer must be a real number!)

We consider the function $f(z) = \frac{\sqrt{z}}{z^4+6}$ on $\mathbb{C} \setminus (-\infty, 0]$, where $\sqrt{z}$ is defined by $\sqrt{|z|}e^{i\frac{\mathrm{Arg}z}{2}}$ with $\mathrm{Arg}z \in (-\pi, \pi)$. Then f is an analytic function on $\mathbb{C} \setminus (-\infty, 0]$.

Then we compute the complex line integral of $f(z)$ on the boundary of

$$D_R = \{z \in \mathbb{C} \mid 0 < |z| < R,\ \mathrm{Re}z > 0,\ \mathrm{Im}z > 0\}.$$

Since the singularity of $f(z)$ in D_R is only $2e^{i\frac{\pi}{4}}$,we have

$$\oint_{\partial D_R} f(z)\mathrm{d}z = 2\pi i \cdot \mathrm{Res}[\frac{\sqrt{z}}{z^4+16}, 2e^{i\frac{\pi}{4}}] = 2\pi i(\frac{\sqrt{z}}{(z^4+16)'})|_{z=2e^{i\frac{\pi}{4}}} = \frac{\sqrt{2}}{16}\pi e^{-i\frac{\pi}{8}}.$$

On the other hand, $\oint_{\partial D_R} f(z)\mathrm{d}z$ consists of three terms: integral along $[0, R]$, integral along the quarter circle γ_R and integral along $[Ri, 0i]$. So we have

$$\oint_{\partial D_R} f(z)\mathrm{d}z = \int_0^R \frac{\sqrt{x}}{x^4+16}\mathrm{d}x + \int_{\gamma_R} \frac{\sqrt{z}}{z^4+16}\mathrm{d}z - \int_0^R \frac{\sqrt{ix}}{(ix)^4+16} \cdot i\mathrm{d}x.$$

By the ML-estimation, we have

$$|\int_{\gamma_R} \frac{\sqrt{z}}{z^4+16}\mathrm{d}z| \leq \frac{\sqrt{R}}{R^4-16} \cdot \frac{\pi}{2}R = \frac{\pi R^{\frac{3}{2}}}{2(R^4-16)}.$$

This term goes to 0 as R goes to ∞.

we let R going to infinity, then

$$\frac{\sqrt{2}}{16}\pi e^{-i\frac{\pi}{8}} = \lim_{R\to\infty} \oint_{\partial D_R} f(z)\mathrm{d}z = \int_0^\infty \frac{\sqrt{x}}{x^4+16}\mathrm{d}x - \int_0^\infty \frac{\sqrt{ix}}{(ix)^4+16} \cdot i\mathrm{d}x = (1-e^{i\frac{3\pi}{4}}) \int_0^\infty \frac{\sqrt{x}}{x^4+16}\mathrm{d}x.$$

So we get

$$\int_0^\infty \frac{\sqrt{x}}{x^4+16}\mathrm{d}x = \frac{\sqrt{2}}{16}\pi \cdot \frac{e^{-i\frac{\pi}{8}}}{1-e^{i\frac{3\pi}{4}}} = \frac{\sqrt{2}}{16}\pi \cdot \frac{1}{e^{i\frac{\pi}{8}} - e^{i\frac{7\pi}{8}}} = \frac{\sqrt{2}\pi}{32\cos\frac{\pi}{8}}.$$

7. (10 points) Please find the number of zeros of the polynomial

$$p(z) = z^7 + iz^5 + 1$$

on the upper half plane $\{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$.

For any $z \in \mathbb{C}$ with $|z| > 2$, we have $|p(z)| \geq |z^7| - |z^5| - 1 \geq (|z^2| - 2) \cdot |z^5| > 0$. So we only need to compute the number of zeros in the half disc

$$D_R = \{z \in \mathbb{C} \mid |z| < R,\ \mathrm{Im} z > 0\},$$

for large R.

For large enough R, we want to define $\arg(p(z))$ continuously on ∂D_R, and check how much does the argument change when we goes along ∂D_R once.

Since R is very large, when z goes from 0 to R on the real axis, $p(z)$ goes from $p(0) = 1$ to $p(R) \approx R^7$. Since $p(z)$ never crosses the y-axis for $z \in [0, R]$, we have $\arg(p(0)) = 0$ and $\arg(p(R)) \approx 0$.

Since $p(-R) \approx -R^7$ and the z^7 term makes the main contribution of $\arg(p(z))$ on the half circle $\{z \in \mathbb{C} \mid |z| = R,\ \mathrm{im} z \geq 0\}$. So we can continuously define $\arg(p(z))$ on the half circle such that $\arg(p(-R)) \approx 7\pi$.

When z goes from $-R$ to 0, $p(z)$ only crosses the y-axis once, since $z^7 + 1$ has only one real solution: $z = -1$. For $z \in [-R, -1)$, $p(z)$ lies in the third quadrant. Since $\arg(p(-R)) \approx 7\pi$ and $p(-1) = -i$, we can continuously defined $\arg(p(z))$ such that $\arg(p(-1)) = 7\pi + \frac{\pi}{2} = \frac{15}{2}\pi$. For $z \in (-1, 0]$, $p(z)$ lies in the fourth quadrant. Since $p(0) = 1$, we end up with $\arg(p(0)) = 8\pi$ when we goes around the boundary of the half disc once.

So $p(z) = z^7 + iz^5 + 1$ has four zeros on the upper half plane.

8. (12 points) Let $f : \{|z| < 1 + \epsilon\} \to \mathbb{C}$ be an analytic function for some $\epsilon > 0$. Suppose that $|f(z)| \leq 1$ for any $z \in \{|z| \leq 1\}$, please follow the following steps and show that there exists $z_0 \in \{|z| \leq 1\}$ such that $f(z_0) = z_0$.

(This is the complex analysis version of the Brouwer fixed point theorem.)

(1) (2 points) For any $n \geq 1$, let $f_n(z) = (1 - \frac{1}{n})f(z)$. Please prove that $\{f_n\}$ uniformly converges to f on $\{|z| \leq 1\}$.

For any $\epsilon > 0$, take $N = [\frac{1}{\epsilon}]$. Then for any $n > N$ and any $z \in \{|z| \leq 1\}$, we have

$$|f_n(z) - f(z)| = \frac{1}{n}|f(z)| \leq \frac{1}{n} < \epsilon.$$

So $\{f_n\}$ uniformly converges to f on $\{|z| \leq 1\}$.

(2) (4 points) For each n, please show that there is a unique $z_n \in \{|z| < 1\}$ such that $f_n(z_n) = z_n$.

On the boundary of the disc $\{|z| = 1\}$, for any n, we have

$$|z| = 1 > 1 - \frac{1}{n} \geq (1 - \frac{1}{n})|f(z)| = |f_n(z)|.$$

So Rouche's theorem implies that $z - f_n(z)$ has the same number of zeros in $\{|z| < 1\}$ as the function z. Since z has a unique zero in $\{|z| < 1\}$, so does $z - f_n(z)$. Suppose this unique zero os z_n, then clearly $f_n(z_n) = z_n$ holds.

(3) (5 points) Suppose that $f(z) \neq z$ for any $z \in \{|z| = 1\}$, please show that there exists a unique $z_0 \in \{|z| < 1\}$ such that $f(z_0) = z_0$. (Hint: you may need to use the sequence of functions $\{f_n\}$.)

Since $f(z) \neq z$ for any $z \in \{|z| = 1\}$, the infimum

$$\delta = \inf\{|f(z) - z| \mid |z| = 1\} > 0.$$

For any $n > \frac{2}{\delta}$ and any $z \in \{|z| = 1\}$, we have

$$|(f_n(z) - z) - (f(z) - z)| = \frac{1}{n}|f(z)| \leq \frac{1}{n} < \frac{\delta}{2} \leq \frac{|f(z) - z|}{2}.$$

In particular, $f_n(z) - z \neq 0$ holds.

Since $\frac{1}{2} = \sin\frac{\pi}{6}$, we can continuously define $\arg(f(z) - z)$ and $\arg(f_n(z) - z)$ on $\{|z| = 1\}$ such that

$$|\arg(f(z) - z) - \arg(f_n(z) - z)| \leq \frac{\pi}{6}.$$

When we go around $\{|z| = 1\}$ once, the amount of the change of $\arg(f(z) - z)$ and $\arg(f_n(z) - z)$ are both integer multiples of 2π, so they are actually equal to each other.

This can be written as the mathematical formula

$$\oint_{|z|=1} \frac{(f(z)-z)'}{f(z)-z}\mathrm{d}z = \oint_{|z|=1} \frac{(f_n(z)-z)'}{f_n(z)-z}\mathrm{d}z.$$

Since $f_n(z) - z$ has a unique zero in $\{|z| < 1\}$, we have $\oint_{|z|=1} \frac{(f_n(z)-z)'}{f_n(z)-z}\mathrm{d}z = 2\pi i$. So $\oint_{|z|=1} \frac{(f(z)-z)'}{f(z)-z}\mathrm{d}z = 2\pi i$ holds, which implies $f(z) - z$ has a unique zero in $\{|z| < 1\}$.

(4) (1 point) Without assuming $f(z) \neq z$ on $\{|z| = 1\}$, please show that there exists $z_0 \in \{|z| \leq 1\}$ such that $f(z_0) = z_0$.

Suppose that there exists $z_0 \in \{|z| = 1\}$ such that $f(z_0) = z_0$, then this is our desired fixed point.

Otherwise, if $f(z) \neq z$ for any $z \in \{|z| = 1\}$, subproblem (3) implies that there exists $z_0 \in \{|z| < 1\} \subset \{|z| \leq 1\}$, such that $f(z_0) = z_0$.