Math 53, First Midterm A

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Instructions: *Please show your work: unjustified answers will not receive credit. Use back of page if needed. (No justification is required in the True/False section, however.) Your signature above certifies that the work here is your own.*

1. Let $f(x, y, z) = ye^{-xz}$. Find the rate of change of f at the point $P = (2, 1, 0)$, in the direction of the point $Q = (3, 2, 1)$.

We have

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle -yze^{-xz}, e^{-xz}, -yxe^{-xz} \rangle.$$

$$\nabla f(2,1,0) = \langle 0, 1, -2 \rangle$$

Let $\vec{u} = \dfrac{1}{|\overrightarrow{PQ}|} \overrightarrow{PQ}$. We want $D_{\vec{u}}(f)(P)$.

But $\overrightarrow{PQ} = \langle 3,2,1 \rangle - \langle 2,1,0 \rangle = \langle 1,1,1 \rangle$

$|\overrightarrow{PQ}| = \sqrt{3}$

$\vec{u} = \dfrac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$

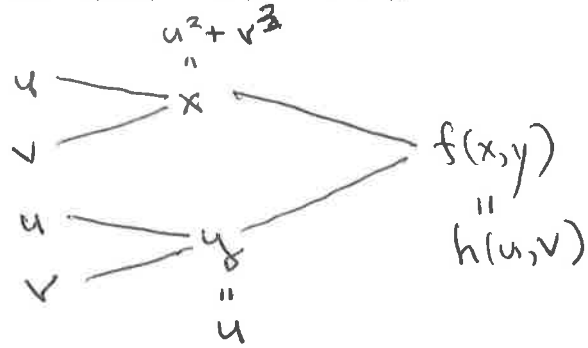
$$D_{\vec{u}}(f)(2,1,0) = \nabla f(2,1,0) \cdot \vec{u}$$

$$= \left(\langle 0, 1, -2 \rangle \cdot \langle 1, 1, 1 \rangle \right) 1/\sqrt{3}$$

$$= -1/\sqrt{3}$$

2. Let $f = f(x, y)$ have continuous partial derivatives. Let $h(u, v) = f(u^2 + v^3, u)$. Compute $\frac{\partial^2 h}{\partial u \partial v}$, in terms of u, v, and the partials of f.

The variable dependence is:



So

$$\frac{\partial h}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

$$= \frac{\partial f}{\partial x} 3v^2 \quad \left(\text{since } \frac{\partial y}{\partial v} = 0\right)$$

So

$$\frac{\partial^2 h}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x} 3v^2\right) = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) 3v^2 + \frac{\partial f}{\partial x}\frac{\partial}{\partial u}(3v^2)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) 3v^2 \quad \left(\text{since } \frac{\partial}{\partial u}(3v^2) = 0\right)$$

$$= \left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial u}\right) 3v^2$$

$$= \left(\frac{\partial^2 f}{\partial x^2} 2u + \frac{\partial^2 f}{\partial y \partial x}\right) 3v^2$$

3. Let $f(x, y) = x^2y - x^2 - 2y^2$. Find the critical points of f, and classify them.

$f_x = 2xy - 2x = 2x(y-1)$

$f_y = x^2 - 4y$

Setting $f_x = 2x(y-1) = 0$, the solutions are $x = 0$, or $y = 1$.

When $x = 0$ $f_y = 0$ iff $0 - 4y = 0$ iff $y = 0$

So $(0,0)$ is a crit. pt.

When $y = 1$ $f_y = 0$ iff $x^2 - 4 = 0$ iff $x = \pm 2$

So $(2,1)$, $(-2,1)$ are the other crit. pts.

Test them: $f_{xx} = 2(y-1)$, $f_{yy} = -4$, $f_{xy} = 2x$

$(0,0)$ $f_{xx}f_{yy} - f_{xy}^2 = (-2)(-8) - 0 = 8 > 0$.

Also $f_{xx} < 0$. So we have a local max.

$(2,1)$ $f_{xx}f_{yy} - f_{xy}^2 = 0 \cdot (-4) - 16 < 0$

$(-2,1)$ $f_{xx}f_{yy} - f_{xy}^2 = 0 \cdot (-4) - 16 < 0$

So $(2,1)$, $(-2,1)$ are saddle points.

4. Let C be the curve of intersection of the surfaces $z = x^2 + y^2$ and $4x^2 + y^2 + z^2 = 9$. Find a nonzero vector that is tangent to C at $(-1, 1, 2)$.

The surfaces are defined implicitly by $f(x,y,z) = 0$, where $f(x,y,z) = x^2 + y^2 - z$, and $g(x,y,z) = 9$, where $g(x,y,z) = 4x^2 + y^2, z^2$.

So

$$\nabla f = \langle 2x, 2y, -1 \rangle$$

and

$$\nabla g = \langle 8x, 2y, 2z \rangle$$

are normals to the surfaces. A vector is tangent to C at $(-1,1,2)$ just in case it is normal to both $\nabla f(-1,1,2)$ and $\nabla g(-1,1,2)$.

So $\nabla f(-1,1,2) \times \nabla g(-1,1,2)$ is such a vector.

$$\nabla f(-1,1,2) \times \nabla g(-1,1,2) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & -1 \\ -8 & 2 & 4 \end{vmatrix}$$

$$= \langle 8-3, 8+16, -20 \rangle$$

$$= \langle 5, 24, -20 \rangle$$

5. **True or False**. (There is no penalty for guessing wrong.)

F 1. For any vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$.

T 2. If $\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$ for all vectors $\vec{c}$, then $\vec{a} = \vec{b}$.

T 3. If a particle moves with constant speed, then its velocity and acceleration vectors are always orthogonal to each other.

F 4. If all partials of F exist and are continuous everywhere, then the equation $F(x, y, z) = F(a, b, c)$ defines a surface near (a, b, c).

F 5. If f has continuous partials of all orders, and $\vec{r}(t)$ has continuous derivatives of all orders, then $\frac{d^2}{dt^2}(f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \frac{d^2\vec{r}}{dt^2}$.

T 6. If $\vec{r}(t)$ describes the motion of a particle whose acceleration vector always points toward the origin, then $\frac{d}{dt}(\vec{r} \times \frac{d\vec{r}}{dt}) = \vec{0}$.

Explanations

(1) $\vec{i} \times (\vec{i} \times \vec{j}) \neq (\vec{i} \times \vec{i}) \times \vec{j}$

(2) We have $\vec{a} \cdot \vec{i} = \vec{b} \cdot \vec{i}$, so $a_1 = a_2$. Also $\vec{a} \cdot \vec{j} = \vec{b} \cdot \vec{j}$, so $a_2 = b_2$. In $\mathbb{R}^3$, we get also $\vec{a} \cdot \vec{k} = \vec{b} \cdot \vec{k}$, so $a_3 = b_3$.

(3) We have $\frac{d}{dt}(\vec{r}\,' \cdot \vec{r}\,') = 0$ by hypothesis. So

$2\vec{r}\,'' \cdot r' = 0$ at all t.

(4) One must assume $\nabla F(a,b,c) \neq \vec{0}$ as well.

(5) This is nonsense, there are many counter-examples.

(6) $\frac{d}{dt}(\vec{r} \times \frac{d\vec{r}}{dt}) = \left(\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt}\right) + \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) = \vec{0} + \vec{0} = \vec{0}$