Math 1B Final Exam Solutions
Lecture 3, Spring 2010

1 • Using a trigonometric substitution, evaluate

$$\int_0^{\sqrt{3}/2} \frac{1}{(1-x^2)^{1/2}} dx$$

Recall that $\sqrt{3}/2 = \sin \pi/3 = \cos \pi/6$.

Solution. Let $x = \sin\theta$, $\theta \in [-\pi/2, \pi/2]$. Then $dx = \cos\theta \, d\theta$. Furthermore, when $x = \sqrt{3}/2$, $\theta = \pi/3$, and when $x = 0$, $\theta = 0$. Thus,

$$\begin{aligned} \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{(1-x^2)^{1/2}} dx &= \int_0^{\frac{\pi}{3}} \frac{1}{\cos\theta} \cos\theta \, d\theta \\ &= \int_0^{\frac{\pi}{3}} d\theta \\ &= \pi/3 \end{aligned}$$

□

2 • Solve the equation

$$t\frac{du}{dt} = u + t^2 \cos t \quad (t > 0)$$

and find a solution that satisfies $u(\pi/2) = 0$.

Solution. We may rewrite the equation as $u' - \frac{1}{t}u = t\cos t$. This is a first order linear differential equation.

$$I(t) = e^{\int -1/t \, dt} = e^{-\ln t} = \frac{1}{t}$$

So our goal is to solve $\left(\frac{u}{t}\right)' = \cos t$. Integrating both sides yields $\frac{u}{t} = \sin t + C$, so

$$u = t\sin t + Ct$$

is the general solution.

Finding the particular solution where $u(\pi/2) = 0$, we get

$$0 = \frac{\pi}{2}\sin\frac{\pi}{2} + C\frac{\pi}{2} = \frac{\pi}{2}(1 + C)$$

so $C = -1$. The equation is then $u = t\sin t - t$. □

3 • Use substitution and integration by parts to find:

$$\int (\cos x)^3 \, e^{\sin x} dx$$

Solution. Let $u = \sin x$. Then $du = \cos x \, dx$. So we obtain

$$\begin{aligned} \int \cos^3 x \, e^{\sin x} dx &= \int (1 - \sin^2 x) e^{\sin x} \cos x \, dx \\ &= \int (1 - u^2) e^u du \\ &= \int e^u du - \int u^2 e^u du \end{aligned}$$

We see that it will be helpful to find $\int x^2 e^x dx$, which can be done using integration by parts twice. First let $u = x^2$ and $dv = e^x$. Then $du = 2x$ and $v = e^x$ and we obtain:

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx$$

The latter integral may be found using a second application of parts. Let $u = x$ and $dv = e^x$. Then $du = 1$ and $v = e^x$, so

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x$$

Thus,

$$\int x^2 e^x dx = x^2 e^x - 2(x e^x - e^x) = e^x (x^2 - 2x + 2)$$

Plugging this result into our main calculation yields

$$\begin{aligned} \int \cos^3 x \, e^{\sin x} dx &= \int e^u du - \int u^2 e^u du \\ &= e^u - e^u (u^2 - 2u + 2) \\ &= -e^u (u^2 - 2u + 1) \\ &= -e^{\sin x} (\sin x - 1)^2 + C \end{aligned}$$

□

4 • Determine if the series absolutely converges, conditionally converges, or diverges.

$$\sum_{n=1}^{\infty} (-1)^n \ln \left(\frac{n}{2n+1} \right)$$

Solution. The limit of the terms of this series is not zero since

$$\left|(-1)^n \ln\left(\frac{n}{2n+1}\right)\right| = \ln\left(\frac{n}{2n+1}\right) \quad \rightarrow \quad \ln(1/2) \neq 0$$

So the series diverges by the Test for Divergence. □

5 • Show that integral

$$\int_1^\infty \frac{e^x}{x+e^{2x}} dx$$

converges or diverges using the comparison test.

Solution. Note that for $x \geq 1$,

$$0 < \frac{e^x}{x+e^{2x}} \leq \frac{e^x}{e^{2x}} = e^{-x}$$

So, if $\int_1^\infty e^{-x}\,dx$ converges, then $\int_1^\infty \frac{e^x}{x+e^{2x}}\,dx$ converges too.

$$\begin{aligned}\int_1^\infty e^{-x}\,dx &= \lim_{t\to\infty} \left[-e^{-x}\right]_1^t \\ &= \lim_{t\to\infty} \left[-e^{-t} + e^{-1}\right] \\ &= e^{-1}\end{aligned}$$

In conclusion, the given integral converges. □

6 • Consider the differential equation

$$\frac{dy}{dx} = \frac{1-x}{2y}$$

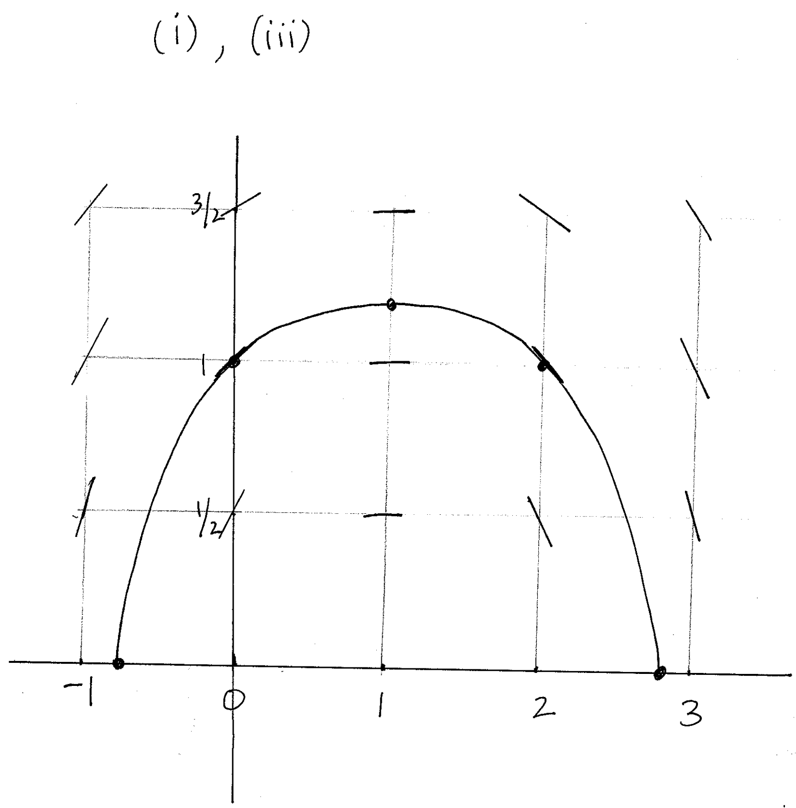
i) Sketch a direction field for the region $-1 \leq x \leq 3$, $0 < y \leq 3/2$, including at least 15 points. Also include labeled axes.

ii) Solve the differential equation. Express y explicitly in terms of x.

iii) Find a solution through $x = 2$, $y = 1$, and sketch it on the direction field graph.

Solution. (i)

x	-1	-1	-1	0	0	0	1	1	1	2	2	2	3	3	3
y	$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$
$\frac{1-x}{2y}$	2	1	$\frac{2}{3}$	1	$\frac{1}{2}$	$\frac{1}{3}$	0	0	0	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	-2	-1	$-\frac{2}{3}$



(ii) We use separation of variables to solve the differential equation. We get $2y\,dy = (1-x)\,dx$. Integrating yields $y^2 = x - \frac{x^2}{2} + C$. Thus, $y = \pm\sqrt{-\frac{x^2}{2} + x + C}$.

(iii) We plug in the values $x = 2$ and $y = 1$ into our equation from part (ii) to find C. We get $1 = \pm\sqrt{-\frac{2^2}{2} + 2 + C} = \pm\sqrt{C}$. Thus $C = 1$ and we choose "+". So $y = \sqrt{-\frac{x^2}{2} + x + 1}$. To aid in making a sketch of this, we might compute its zeros. These occur just when $-\frac{x^2}{2} + x + 1 = 0$, or

$$x = \frac{-1 \pm \sqrt{1 - 4(-1/2)(1)}}{2(-1/2)} = \frac{-1 \pm \sqrt{3}}{-1} = 1 \pm \sqrt{3} \approx -0.7, 2.7$$

Also, we note that $(0,1)$ and $(2,1)$ are on the curve. Finally, the direction field can give us a sense of how the curve may be filled in (see figure above). □

7 • Find the integral

$$\int_0^2 \frac{x}{(x^2-1)^2}dx,$$

if it converges. If it does not converge, show why that happens.

Solution. Note that the denominator equals 0 when $x^2 - 1 = 0$, or $x = \pm 1$. So there is one discontinuity between the limits of integration, namely at $x = 1$. Then

$$\int_0^2 \frac{x}{(x^2-1)^2}dx = \lim_{t\to 1^-}\int_0^t \frac{x}{(x^2-1)^2}dx + \lim_{t\to 1^+}\int_t^2 \frac{x}{(x^2-1)^2}dx$$

provided that both limits exist. Otherwise, the integral diverges. To determine whether these limits exist, it will be useful to find the indefinite integral $\int \frac{x}{(x^2-1)^2}\,dx$, which can be done using substitution. Let $u = x^2 - 1$. Then $\frac{du}{2} = x\,dx$, and we obtain:

$$\begin{aligned}\int \frac{x}{(x^2-1)^2}\,dx &= \int \frac{1}{u^2}\frac{du}{2} \\ &= \frac{-1}{2u} \\ &= \frac{-1}{2(x^2-1)} + C\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{t\to 1^-}\int_0^t \frac{x}{(x^2-1)^2}\,dx &= \lim_{t\to 1^-}\left[\frac{-1}{2(t^2-1)} + \frac{1}{2(0^2-1)}\right] \\ &= \infty\end{aligned}$$

So, the integral diverges. □

8 • Find the Taylor series for

$$\frac{1}{2}x^2(e^x - e^{-x})$$

around $x = 0$. What is the coefficient of x^n? What is its radius of convergence?

Solution. Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ with radius of convergence $R = \infty$. Thus,

$$\begin{aligned}
\frac{1}{2}x^2\left(e^x - e^{-x}\right) &= \frac{1}{2}x^2\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right) \\
&= \frac{1}{2}x^2\sum_{n=0}^{\infty}\frac{x^n - (-1)^n x^n}{n!} \\
&= \frac{1}{2}x^2\sum_{n=0}^{\infty}\frac{2x^{2n+1}}{n!} \\
&= \sum_{n=0}^{\infty}\frac{x^{2n+3}}{n!} \\
&= \frac{x^3}{0!} + \frac{x^5}{1!} + \frac{x^7}{2!} + \frac{x^9}{3!} + \cdots
\end{aligned}$$

The coefficient of x^n is 0 if n is even or 1, and otherwise it's given by $1/\left(\frac{n-3}{2}\right)!$. The radius of convergence is $R = \infty$, as the radius of convergence for the e^x series is ∞. □

9 • Consider the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+1}$.

i) Use the limit comparison test to show that the series is convergent.

ii) Determine whether

$$0 \le \sum_{n=1}^{\infty}\frac{1}{n^2+1} \le \frac{\pi}{2}$$

is true or false by comparing the series to an integral from 0 to infinity.

Solution. (i) The series $\sum \frac{1}{n^2}$ is a convergent p-series. The series $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^2+1}$ are both series with positive terms, and

$$\lim_{n\to\infty}\frac{\frac{1}{n^2}}{\frac{1}{n^2+1}} = \lim_{n\to\infty}\frac{n^2+1}{n^2} = \lim_{n\to\infty} 1 + \frac{1}{n^2} = 1$$

Thus, the limit comparison test guarantees that $\sum_{n=0}^{\infty}\frac{1}{n^2+1}$ converges too.

(ii) True. Since $\frac{1}{x^2+1}$ is a continuous, positive, decreasing function on $[0, \infty)$,

$$0 \le \sum_{n=1}^{\infty}\frac{1}{n^2+1} \le \int_0^{\infty}\frac{dx}{x^2+1}$$

But this integral is simply:

$$\begin{aligned}\int_0^\infty \frac{dx}{x^2+1} &= \lim_{t\to\infty}\int_0^t \frac{dx}{x^2+1}\\ &= \lim_{t\to\infty}\left[\tan^{-1}(x)\right]_0^t\\ &= \pi/2\end{aligned}$$

Thus $0 \le \sum_{n=1}^{\infty} \frac{1}{n^2+1} \le \frac{\pi}{2}$. □

10 • Find the Maclaurin series for

$$x\cos x - \sin x.$$

and use that to find the limit

$$\lim_{x\to 0}\frac{x\cos x - \sin x}{x^3}$$

Solution. Recall that $\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$ and $\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Thus,

$$\begin{aligned}x\cos x - \sin x &= \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+1}}{(2n)!} - \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right]\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)x^{2n+1} - (-1)^n x^{2n+1}}{(2n+1)!}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}(2n+1-1)}{(2n+1)!}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(2n)x^{2n+1}}{(2n+1)!}\end{aligned}$$

So

$$\begin{aligned}\frac{x\cos x - \sin x}{x^3} &= \sum_{n=0}^{\infty}\frac{(-1)^n(2n)x^{2n-2}}{(2n+1)!}\\ &= 0 + \frac{(-1)(2)x^0}{3!} + \frac{(1)(2\cdot 2)x^2}{5!} + \cdots\end{aligned}$$

As power series are continuous in their intervals of convergence, and this one has $R=\infty$, we see that

$$\begin{aligned}\lim_{x\to 0}\frac{x\cos x - \sin x}{x^3} &= \frac{-2}{3!} + 0 + 0 + \cdots\\ &= -\frac{1}{3}\end{aligned}$$

□

11 • Find the radius and the interval of convergence of the power series

$$\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{2^n \ln n}$$

Solution. First we use the Ratio Test to determine the radius of convergence.

$$\left|\frac{(-1)^{n+1}x^{n+1}}{2^{n+1}\ln(n+1)} \cdot \frac{2^n \ln n}{(-1)^n x^n}\right| = \frac{1}{2}\frac{\ln n}{\ln(n+1)}|x| \ \rightarrow \ \frac{1}{2}|x|$$

Thus, $R = 2$.

To determine the interval of convergence, we must check the endpoints $x = -2$ and $x = 2$. First, consider $x = -2$. The series becomes

$$\sum_{n=2}^{\infty} \frac{(-1)^n(-2)^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

which diverges by the Comparison Test, since $0 < \frac{1}{n} \leq \frac{1}{\ln n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges. Now consider $x = 2$. The series becomes

$$\sum_{n=2}^{\infty} \frac{(-1)^n 2^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

which converges by the Alternating Series Test, since $\frac{1}{\ln n}$ is a decreasing positive sequence with limit 0. In conclusion, $I = (-2, 2]$. □

12 • Use the power series method to find the general solution to

$$y'' = x^2 y$$

Then, find the solution to the initial value problem $y(0) = 1, y'(0) = 5$.

Solution. Suppose $y = \sum_{n=0}^{\infty} c_n x^n$. Then

$$\begin{aligned} y'' - x^2 y &= \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=0}^{\infty} c_n x^{n+2} \\ &= 2c_2 + 6c_3 x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)c_{n+2} - c_{n-2}\right]x^n \\ &= 0 \end{aligned}$$

So

$$
\begin{aligned}
c_2 &= 0 \\
c_3 &= 0 \\
c_4 &= \frac{c_0}{4 \cdot 3} \\
c_5 &= \frac{c_1}{5 \cdot 4} \\
c_6 &= 0 \\
c_7 &= 0 \\
c_8 &= \frac{c_4}{8 \cdot 7} = \frac{c_0}{8 \cdot 7 \cdot 4 \cdot 3} \\
c_9 &= \frac{c_5}{9 \cdot 8} = \frac{c_1}{9 \cdot 8 \cdot 5 \cdot 4} \\
&\cdots
\end{aligned}
$$

Thus

$$y = c_0 \left(1 + \sum_{n=1}^{\infty} \frac{x^{4n}}{(4n)(4n-1) \cdot \cdots \cdot 8 \cdot 7 \cdot 4 \cdot 3}\right) + c_1 \left(x + \sum_{n=1}^{\infty} \frac{x^{4n+1}}{(4n+1)(4n) \cdot \cdots \cdot 9 \cdot 8 \cdot 5 \cdot 4}\right)$$

If $y(0) = 1$, then $c_0 = 1$, and if $y'(0) = 5$, then $c_1 = 5$. In this case we get

$$y = \left(1 + \sum_{n=1}^{\infty} \frac{x^{4n}}{(4n)(4n-1) \cdot \cdots \cdot 8 \cdot 7 \cdot 4 \cdot 3}\right) + 5 \left(x + \sum_{n=1}^{\infty} \frac{x^{4n+1}}{(4n+1)(4n) \cdot \cdots \cdot 9 \cdot 8 \cdot 5 \cdot 4}\right)$$

□

13 • Solve the differential equation

$$y'' + y' - 6y = 4x^2 e^x + e^{-x}$$

Solution. First we solve the corresponding homogenous equation $y'' + y' - 6y = 0$

$$
\begin{gathered}
r^2 + r - 6 = 0 \\
(r+3)(r-2) = 0 \\
r = -3 \text{ or } r = 2 \\
y_c = c_1 e^{-3x} + c_2 e^{2x}
\end{gathered}
$$

This doesn't conflict with $4x^2 e^x + e^{-x}$, so we may try

$$y_p = \left(Ax^2 + Bx + C\right) e^x + De^{-x}$$

Then

$$
\begin{aligned}
y_p' &= (2Ax+B)\,e^x + \left(Ax^2+Bx+C\right)e^x + -De^{-x} \\
&= \left(Ax^2+[2A+B]x+[B+C]\right)e^x + -De^{-x} \\
y_p'' &= (2Ax+[2A+B])\,e^x + \left(Ax^2+[2A+B]x+[B+C]\right)e^x + De^{-x} \\
&= \left(Ax^2+[4A+B]x+[2A+2B+C]\right)e^x + De^{-x} \\
y_p''+y_p'-6y_p &= ([A+A-6A]x^2+[4A+B+2A+B-6B]x \\
&\quad +[2A+2B+C+B+C-6C])e^x + [D-D-6D]e^{-x} \\
&= \left(-4Ax^2+[6A-4B]x+[2A+3B-4C]\right)e^x + -6De^{-x} \\
&= 4x^2e^x + e^{-x}
\end{aligned}
$$

It follows that the following equations should be satisfied.

$$
\begin{cases}
-4A = 4 \\
6A - 4B = 0 \\
2A + 3B - 4C = 0 \\
-6D = 1
\end{cases}
$$

From the first of these we get $A = -1$. Then from the second we get $-6 - 4B = 0$, so $B = -\frac{6}{4} = -\frac{3}{2}$. Now, from the third equation, we may obtain $-2 - \frac{9}{2} - 4C = 0$, so that $-4C = \frac{4}{2} + \frac{9}{2}$, and $C = -\frac{13}{8}$. Finally, from the fourth equation we obtain $D = -\frac{1}{6}$. Thus,

$$
y_p = \left(-x^2 - \frac{3}{2}x - \frac{13}{8}\right)e^x - \frac{1}{6}e^{-x}
$$

So the general solution is

$$
y = y_c + y_p = c_1e^{-3x} + c_2e^{2x} + \left(-x^2 - \frac{3}{2}x - \frac{13}{8}\right)e^x - \frac{1}{6}e^{-x}
$$

□