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Fall 2005, Math 55
Solutions to the **First Midterm**

19 Sept., 2005
3:10-4:00

1. (30 points, 10 points each.) Short answer questions. A correct answer will get full credit whether or not work is shown. An incorrect answer may get partial credit if work is given that follows a basically correct method.

(a) Give the truth table for the proposition $(q \to p) \wedge p$. (Your table must show columns for p, for q and for this compound proposition. It may or may not have other columns.)

Answer: (The column $q \to p$ is not required, but is useful in computing the last column.)

p	q	$q \to p$	$(q \to p) \wedge p$
T	T	T	T
T	F	T	T
F	T	F	F
F	F	T	F

(b) Let $f: \mathbf{R} \to \mathbf{R}$ be given by the rule $f(x) = 3x+1$, and $g: \mathbf{R} \to \mathbf{R}$ by the rule $g(x) = 4x+1$. Then $f \circ g: \mathbf{R} \to \mathbf{R}$ is given by the rule

Answer: $(f \circ g)(x) = 12x+4$.

(c) Write in mathematical symbols the statement that every real number which is not an integer lies between some integer and its successor (where the successor of an integer n means the integer $n+1$. If your statement is long, you don't have to put it all on one line.)

Answer: $\forall\, x \in \mathbf{R} - \mathbf{Z}\ \exists\, n \in \mathbf{Z}\ (n < x < n+1)$. *Variants are possible, for instance,* $\forall\, x\, ((x \in \mathbf{R}) \wedge \neg(x \in \mathbf{Z})) \to (\exists\, n ((n \in \mathbf{Z}) \wedge (n < x < n+1)))$.

2. (24 points, 8 points each.) Complete the following definitions. Your definitions do not have to have exactly the same wording as those in the text, but for full credit they should be clear, and be equivalent to those.

(a) If X and Y are sets, and $f: X \to Y$ is a function, then the *graph* of f is defined to be the set ...

Answer: $\{(x,y) \mid x \in X,\ y = f(x)\}$. *(Also OK:* $\{(x, f(x)) \mid x \in X\}$.)

(b) Let I be a set, and suppose that for each $i \in I$ we are given a set A_i. Then $\cup_{i \in I} A_i$ denotes ... (For full credit, use set-builder notation rather than words.)

Answer: $\{x \mid \exists\, i \in I\, (x \in A_i)\}$.

(c) If $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ are functions, then one says that $f(x)$ is $\Omega(g(x))$ (in words, "$f(x)$ is big-Omega of $g(x)$") if ...

Answer: $\exists\, k>0\ \exists\, C>0\ \forall\, x>k\ (|f(x)| \geq C|g(x)|)$.

3. (30 points, 15 points each.) Short proofs. In giving the proofs asked for below, you may call upon definitions and results proved or asserted in the text. You do not have to use the formal names of methods of proof, or any standardized format, as long as your arguments are clear and logically sound.

(a) Suppose X, Y and Z are sets, and $f: X \to Y$ a function. Prove that $f(X-Z) \supseteq f(X) - f(Z)$.

Answer: To show this inclusion, we must prove that every element of $f(X) - f(Z)$ belongs to $f(X-Z)$. If $a \in f(X) - f(Z)$, that means $a \in f(X)$ but $a \notin f(Z)$. Since $a \in f(X)$, we can find some $b \in X$ such that $a = f(b)$. Now if b were a member of Z, we would have $f(b) \in f(Z)$, i.e., $a \in f(Z)$, which we have just noted is not true. Hence $b \notin Z$, so since $b \in X$ we have $b \in X-Z$. Since $a = f(b)$, this says that $a \in f(X-Z)$, as required.

(b) Suppose that $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ are functions such that $f(x)$ is $O(g(x))$. Prove that $f(\log(|x|))$ is $O(g(\log(|x|)))$. (Recall that in this course, "log" means the logarithm to the base 2. You may use well-known facts about the logarithm function.)

Answer: By the definition of the statement that $f(x)$ is $O(g(x))$, there exist real constants C and k ("witnesses" to the big-O relation) such that for all $x > k$ one has $|f(x)| \leq C\,|g(x)|$. I claim that for all $x > 2^k$ one has $|f(\log(|x|))| \leq C\,|g(\log(|x|))|$. Indeed, if $x > 2^k$ then $\log(|x|) = \log(x) > \log 2^k = k$, so the inequality for f holds with $\log x$ in the role of x, giving the asserted inequality. Hence C and 2^k are witnesses to the statement that $f(\log(|x|))$ is $O(g(\log(|x|)))$.

4. (16 points) Write in pseudocode an algorithm "*intersect*" which takes two sequences of real numbers $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$, where the elements of each sequence are distinct (i.e., for $i \neq j$, one has $a_i \neq a_j$ and $b_i \neq b_j$) and creates a sequence $c_1, \ldots, c_r$ of distinct real numbers such that

$$\{c_1, \ldots, c_r\} = \{a_1, \ldots, a_m\} \cap \{b_1, \ldots, b_n\}.$$

That is, after the algorithm has run, r should equal the number of elements in that intersection, and $c_1, \ldots, c_r$ should be the distinct elements in that intersection. For full credit you should use only the basic operations given in the text, and follow the format for pseudocode specified there. (However, you are not expected to give any equivalent of the changes between **bold** *italic* and roman font that the text uses.)

Answer: I will use fonts as in the text, though as stated, your exams are not expected to show anything similar. Here is one solution:

```
procedure intersect(a_1, ..., a_m : distinct real numbers, b_1, ..., b_n : distinct real numbers)
r := 0
for i := 1 to m
begin
    for j := 1 to n
    begin
        if a_i = b_j
        begin
            r := r+1
            c_r := a_i
        end
    end
end
```

☛ Reminder: The reading for Wednesday is #7 ☚