George M. Bergman | Fall 2005, Math 55 | 31 October, 2005

100 Lewis Hall | Solutions to the **Second Midterm** | 1:10-2:00

1. (24 points, 8 points each.) Short answer questions. A correct answer will get full credit whether or not work is shown. An incorrect answer may get partial credit if work is given that follows a basically correct method.

(a) How many 8-element subsets does a 100-element set have? You may express your answer in any of the forms used in the text.

Answer: Acceptable forms of the answer include: $C(100,8)$, $\binom{100}{8}$, $100!/(8!\,(100-8)!)$ *and* $(100\cdot 99\cdot 98\cdot 97\cdot 96\cdot 95\cdot 94\cdot 93)/(1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8)$.

(b) In how many ways can one put together a packet of 30 pieces of fruit, if one has 5 kinds of fruit in unlimited supply, and all that matters is how many of each kind go into the packet? Again, you may express your answer in any of the forms used in the text.

Answer: Acceptable forms include: $C(34,4)$, $\binom{34}{4}$, $34!/(4!\,30!)$ $(34\cdot 33\cdot 32\cdot 31)/(4\cdot 3\cdot 2\cdot 1)$ *and* 46,376.

(c) What is the probability that an integer chosen at random from $\{0,1,\ldots,10\}$ (where all members of this set have equal probability of being chosen) is odd? (Note: Do not confuse $\{0,1,\ldots,10\}$ with $\{1,\ldots,10\}$.)

Answer: $5/11$.

2. (24 points, 8 points each.) Complete the following definitions. Your definitions do not have to have the same wording as those in the text, but for full credit they should be clear, and be equivalent in meaning to those.

(a) Integers m and n are said to be *relatively prime* if . . . *Answer:* $\gcd(m,n)=1$.

(b) A set S is said to be *countable* if . . .

Answer: S is either finite, or has the cardinality of the set of positive integers.

(c) The *lexicographic order* on the set of length-n strings of integers is defined by considering a string $(a_1,\ldots,a_n)$ to precede a string $(b_1,\ldots,b_n)$ under this order if . . .

Answer: for some $k\in\{1,\ldots,n\}$ *one has* $a_1=b_1,\ \ldots,\ a_{k-1}=b_{k-1}$, *but* $a_k<b_k$.
(*Equivalently: the strings are distinct, and the least* k *such that* $a_k\neq b_k$ *satisfies* $a_k<b_k$.)

3. (24 points, 8 points each.) For each of the items listed below, either *give an example* with the properties stated, or give a brief reason why *no such example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, you should name a particular one. If you give a reason why no example exists, don't worry about giving reasons for your reasons; a simple statement will suffice.

(a) Two integers a and b, neither of which is divisible by 17, such that $a^{16}\not\equiv b^{16} \pmod{17}$.

Answer: Not possible. By Fermat's Little Theorem, a^{16} *and* b^{16} *will both be* $\equiv 1 \pmod{17}$, *hence they will be congruent to each other.*

(b) A program which never halts. (Use pseudocode if you give an example.)

Answer: Many possibilities. A simple one is

```
procedure neverhalt
while 1 = 1
        x := 1
```

(c) A one-to-one function from the set of ordered pairs (a,b) with $a,b\in\{1,\ldots,5\}$ to the set of ordered pairs (c,d) with $c\in\{1,\ldots,6\}$, $d\in\{1,\ldots,4\}$.

Answer: Does not exist. The domain has $5\cdot 5=25$ *elements and the codomain has* $6\cdot 4=24$, *so by the pigeonhole principal, some pair of elements of the domain must map to the same element of the codomain.*

4. (28 points, 14 points each.) Short proofs. I am giving you a page for each, in case some of you give roundabout proofs or have several false starts. But concise proofs should take less than half a page each.

(a) Show that if a, b, A, B are integers and m a positive integer, with $a \equiv A \pmod{m}$ and $b \equiv B \pmod{m}$, then $ab \equiv AB \pmod{m}$. Since this is a result proved in Rosen (though with slightly different notation), you may not call on that result, or results proved from it, in your proof of this statement.

Answer: See Rosen, p.163, proof of Theorem 10. (He writes a, b, c, d *where I used* a, A, b, B.*)*

(b) Prove that for every nonnegative integer n, one has $\Sigma_{i=0}^{n}\ i\,2^{i} = (n-1)\,2^{n+1} + 2$.

Answer: We use induction on n.

In the base case, $n = 0$, *the left-hand side is the sum of the single term* $0 \cdot 2^{0} = 0 \cdot 1 = 0$, *while the right-hand side is* $(0-1) \cdot 2^{0+1} + 2 = (-1) \cdot 2 + 2 = 0$, *so they are indeed equal.*

Now assume the result true for $n = k$:

$$\Sigma_{i=0}^{k}\ i\,2^{i} = (k-1)\,2^{k+1} + 2.$$

Then the $n = k+1$ *case, which we need to prove, has on the left-hand side*

$$\Sigma_{i=0}^{k+1}\ i\,2^{i}$$

or, separating off the $i = k+1$ *summand,*

$$(\Sigma_{i=0}^{k}\ i\,2^{i}) + (k+1)\,2^{k+1}.$$

When we apply our inductive hypothesis to the sum up to $i = k$, *this becomes*

$$\begin{aligned} & ((k-1)\,2^{k+1} + 2) + (k+1)\,2^{k+1} \\ =\ & (k-1 + k+1)\,2^{k+1} + 2 \\ =\ & 2\,k\,2^{k+1} + 2 \\ =\ & k\,2^{k+2} + 2, \end{aligned}$$

which is the desired right-hand side. Thus, the inductive step is proved, completing the induction.

☛ Reminder: The reading for Wednesday is #23 ☚