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Math 54, Spring 2015 Professor Mariusz Wodzicki

Discussion Section #

203

Midterm 2

April 24, 2015

Problem	1	2	3	4	5	Total
Score	0	4	5	1	3	13/18

1. Calculate $\cos(At)$ where

$$A := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Provide a detailed calculation. Your answer must be an *explicit* 3×3 matrix with ij-entries being functions of t.

$$\cos(At) = \cos\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} t\right) = \begin{bmatrix} \cos(0t) & \cos(t) & \cos(0t) \\ \cos(0t) & \cos(0t) & \cos(2t) \\ \cos(0t) & \cos(0t) & \cos(0t) \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1 & \cos t & 1 \\ 1 & 1 & \cos 2t \\ 1 & 1 & 1 \end{bmatrix}}$$

X

2. Consider the inner product $\langle f,g\rangle := \int_0^1 f(t)g(t)dt$ on the vector space $C[0,1]$ of continuous real valued functions on the interval $[0,1]$. Find the *orthogonal projection* $\hat{p}$ of $p = t^2$ on the vector space spanned by functions 1 and t. Provide a detailed calculation. Your answer must be in the form $a+bt$ where a and b are explicit real numbers.

can only orthogonally project onto orthogonal functions. Make t orthogonal to 1 w/ gram Schmidt:

$V_1 = 1$

$$V_2 = t - \frac{\langle 1,t\rangle}{\langle 1,1\rangle} 1 = t - \frac{\int_0^1 t\,dt}{\int_0^1 dt} 1 = t - \frac{\frac{1}{2}t^2|_0^1}{t|_0^1} = t - \frac{1}{2} = V_2$$

Now $\hat{p} = \frac{\langle P, V_1\rangle}{\langle V_1, V_1\rangle} V_1 + \frac{\langle P, V_2\rangle}{\langle V_2, V_2\rangle} V_2$, $P = t^2$

$$\hat{p} = \frac{\int_0^1 t^2 dt}{\int_0^1 dt} 1 + \frac{\int_0^1 t^2(t-\frac{1}{2})dt}{\int_0^1 (t-\frac{1}{2})^2 dt}(t-\tfrac{1}{2})$$

$$= \frac{\frac{1}{3}t^3|_0^1}{t|_0^1} + \frac{\int_0^1 t^3 - \frac{1}{2}t^2 dt}{\int_0^1 t^2 - t + \frac{1}{4} dt}(t-\tfrac{1}{2})$$

$$= \frac{\frac{1}{3}}{1} + \frac{\frac{1}{4}t^4 - \frac{1}{6}t^3|_0^1}{\frac{1}{3}t^3 - \frac{1}{2}t^2 + \frac{1}{4}t|_0^1}(t-\tfrac{1}{2}) = \frac{1}{3} + \frac{\frac{1}{4} - \frac{1}{6}}{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}}(t-\tfrac{1}{2})$$

$$= \frac{1}{3} + \frac{\frac{1}{12}}{\frac{1}{12}}(t-\tfrac{1}{2}) = \frac{1}{3} + t - \frac{1}{2} = t + \frac{2}{6} - \frac{3}{6} = t - \frac{1}{6}$$

$$\boxed{\hat{P} = t - \frac{1}{6}}$$

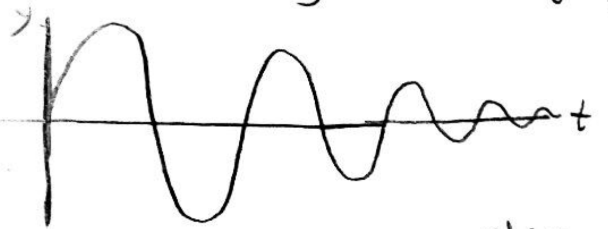
5

3. Consider the differential equation $y'' + by' + cy = 0$ such that $b^2 < 4c$. For which values of b *every* solution is bounded on the interval $[0,\infty)$ and for which values of b one has $\lim_{t\to\infty} y(t) = 0$? Illustrate your answers by drawing the graphs of typical nonzero solutions. What happens for other values of b?

For quadratic formula: $\dfrac{-b \pm \sqrt{b^2-4AC}}{2A} \overset{A=1}{=} -\dfrac{b}{2} \pm \dfrac{\sqrt{b^2-4C}}{2}$

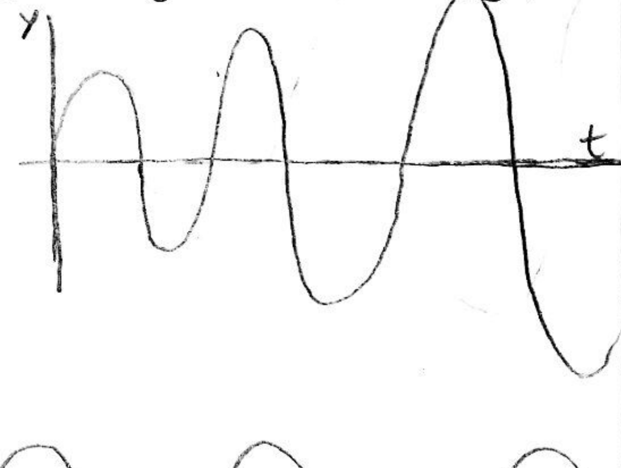
Let's call $-\frac{b}{2} = \alpha$ and $\frac{\sqrt{b^2-4c}}{2} = \beta$. The solution to this second order differential equation is in the form of (since it has imaginary roots) $y = C_1 e^{\alpha t}\cos\beta t + C_2 e^{\alpha t}\sin\beta t$. β has no effect on the solution as $t \to \infty$ since it will simply oscilate; only α will dictate asymptotic behavior.

when $\alpha = -\frac{b}{2} < 0$, or when $b > 0$, $e^{\alpha t}$ will go to zero asymtotically, and the graph of the solution would be (see left).

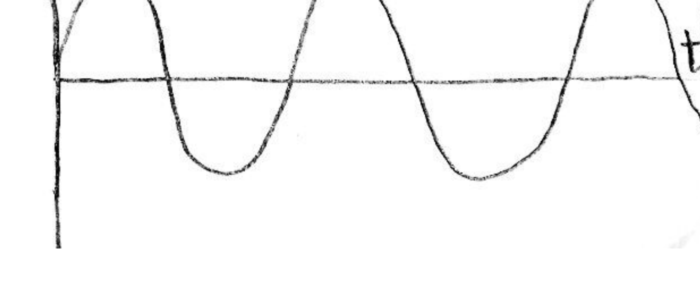


when $b > 0$ is when every solution is bounded on the interval $[0, \infty)$. Here, $\lim_{t\to\infty} y(t) = 0$.

when $\alpha = -\frac{b}{2} > 0$, or when $b < 0$, $e^{\alpha t}$ will go to ∞ and $-\infty$ as $t \to \infty$ since $e^{\alpha t}$ will grow expoentially, and so solutions will NOT be bounded when $b < 0$.



when $\alpha = -\frac{b}{2} = 0$, or when $b = 0$, $e^{\alpha t} = 1$ and the graph will be periodically sinosoidal. Here, every solution is still bounded on the interval $[0,\infty)$, but the $\lim_{t\to\infty} \neq 0$. The limit would bounce between minimum and maximum of the function.



4. Express the solution of the initial value problem

$$\begin{cases} \mathbf{x}'(t) = A\mathbf{x}(t), \\ \mathbf{x}(t_0) = \mathbf{v} \end{cases}$$

X = fund matrix = $[x_1\ x_2\ x_3]$

in terms of the matrix valued function e^{At}. Here $\mathbf{v}$ is a given vector in $\mathbf{R}^n$. Provide a detailed explanation.

We use the definition of e^{At};

$\tilde{X}\tilde{X}^{-1}(t_0) = e^{At}$, where $\tilde{X}$ is the fundamental matrix

if $x(t)$ is the solution to the initial valued problem,

then $x(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, and $\tilde{X} = [x(t)\ x_2(t) \ldots x_n(t)]$

It is given that $x(t_0) = V$ and $x'(t) = Ax(t)$, so then

$\Rightarrow x'(t_0) = Ax(t_0) = AV$, where $V = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$

Then, manipulating the top equation:

$\tilde{X}\tilde{X}^{-1}(t_0) = e^{At}$

$\tilde{X} = e^{At}\tilde{X}(t_0)$, we can express $\tilde{X}(t_0)$ as the solution $x(t)$:

$\tilde{X}(t_0) = [x_1(t_0)\ x_2(t_0) \ldots x_n(t_0)] = [V_1\ V_2 \ldots V_n]$

$\tilde{X} = [x_1(t)\ x_2(t) \ldots x_n(t)] = e^{At}\tilde{X}(t_0) = e^{At}[V_1\ V_2 \ldots V_n]$

$[x_1(t)\ x_2(t) \ldots x_n(t)] = e^{At}[V_1\ V_2, \ldots V_n]$

since V is a vector, $[V_1\ V_2, \ldots V_n]$ is V^T

so $\boxed{x(t) = e^{At}V}$

5. For a continuous function $f(t)$ on the interval $[-L, L]$ let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right\}$$

be its Fourier series. Suppose that all a_i-coefficients are zero. Show that all b_i-coefficients in the Fourier series for $f^2(t)$ are zero.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L}\int_{-L}^{L} f^2(t) \sin\left(\frac{n\pi t}{L}\right) dt$$

$f^2(t)$ here is an even function. This is because
why?
$f^2(-t) = f^2(t)$. $\sin\left(\frac{n\pi t}{L}\right)$ here is an odd function. This is because $\sin\left(\frac{-n\pi t}{L}\right) = -\sin\left(\frac{n\pi t}{L}\right)$. When an even and an odd function multiply, the result is an odd function.

This can be shown with functions f and g where f is odd and g is even:

$f(-t)g(-t) = (-f(t))g(t) = -f(t)g(t)$.

Therefore, $f^2(t)\sin\left(\frac{n\pi t}{L}\right)$ is an odd function, and the integral of an odd function from $-L$ to L is 0.

Therefore:

$$b_i = \sum b_n = \sum \frac{1}{L}\int_{-L}^{L} f^2(t) \sin\left(\frac{n\pi t}{L}\right) = \sum 0 = \boxed{0}$$