1. (10 points) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Find a 3 by 2 matrix B such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution: Writing $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ and multiplying gives the system:

$$\begin{aligned} a + c + e &= 1 \\ c &= 0 \\ b + d + f &= 0 \\ d &= 1 \end{aligned}$$

Thus we can choose for example $a = 1$, $b = -1$, $c = 0$, $d = 1$, $e = 0$, and $f = 0$.

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

(a) (3 points) Find the determinant of A.

Solution: $\text{Det}(A) = 9$.

(b) (4 points) Is A invertible? If so, find A^{-1}.

Solution: Yes. $A^{-1} = \begin{bmatrix} 1/3 & -1/3 & -1/3 & 2/3 \\ 1/9 & 8/9 & 11/9 & -10/9 \\ -2/9 & 2/9 & 5/9 & 2/9 \\ -1/9 & 1/9 & -2/9 & 1/9 \end{bmatrix}$

(c) (3 points) Use parts a and b to find all solutions to $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

Solution: There exists a unique solution

$$\vec{x} = A^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 10/9 \\ 25/9 \\ -1/9 \end{bmatrix}$$

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3. (10 points) Label the following statements as either true or false.

   (a) (2 points) Every matrix can be row reduced into a unique row echelon form.

   (b) (2 points) If A is a square matrix with $\det(A) = 2$, then $\det(A^{-1}) = \frac{1}{2}$.

   (c) (2 points) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is an injective linear transformation, then $n < m$.

   (d) (2 points) Any collection of k vectors in $\mathbb{R}^n$ containing the zero vector is a linearly dependent set.

   (e) (2 points) Let A be an n by n matrix such that $A^2 = A$, then A is invertible.

Solution: a) This is false, only reduced row echelon form is unique. For example, the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is already in row echelon form, but can also be row reduced into $A' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, which is a different row echelon form.

b) This is true because $\det(A)\det(A^{-1}) = \det(I) = 1$, so we have $2 \cdot \det(A^{-1}) = 1$, so $\det(A^{-1}) = \frac{1}{2}$.

c) This is false. For example let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity transformation. That is, $T(v) = v$. This is clearly injective, but we have $n = m$ in this case.

d) This is true. If our vectors are $\vec{0}, v_2, \ldots v_k$, then we have:

$$1 \cdot \vec{0} + 0 \cdot v_2 + \ldots 0 \cdot v_k = \vec{0}$$

So our set is linearly dependent.

e) This is false, for example A could be the zero matrix.

4. (10 points) Consider the subspace U of $\mathbb{R}^5$ spanned by the set of vectors below.

$$\left\{ \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-5\\0\\5\\0\end{pmatrix} \right\}$$

Compute the dimension of this subspace. What is a basis for U?

Solution: We need to determine the maximum number of vectors in this set that are linearly independent. We can immediately remove the zero vector and the last vector since the last vector is -5 times the third vector. Let's check if the remaining three vectors are linearly independent. We have

$$\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\\1&-1&0\\1&0&0\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&-1&0\\0&0&0\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

which has a pivot in every column. We thus conclude that $\dim(U) = 3$ and that

$$\left\{ \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} \right\}$$

is a basis for U.

5. (10 points) (a) (2 points) State the rank theorem for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

> **Solution:** It states: $\dim(\text{Range}(T)) + \dim(\text{Ker}(T)) = n$.

(b) (3 points) Compute the rank of

$$A = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 3 & -1 & 1 & 2 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

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> **Solution:** Row reducing yields 3 pivot columns, so the rank is 3.

(c) (3 points) Is the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(v) = Av$ injective? It may be helpful to use the previous parts.

> **Solution:** We have $\dim(\text{Ker}(T)) = 4 - \dim(\text{Range}(T) = 4 - \text{Rank}(A) = 1$. So we see that T has a nontrivial kernel, and is therefore not injective.

(d) (2 points) Use the rank theorem to show that any linear map from $\mathbb{R}^n \to \mathbb{R}^m$ cannot be injective (one-to-one) if $n > m$.

> **Solution:** We have $\dim(\text{Ker}(T)) = n - \dim(\text{Range}(T) = n - \text{Rank}(A) \geq n - m > 0$, so T is not injective.

6. (10 points) For each of the following provide an explicit example:

(a) (3 points) A matrix A with $A^2 = 0$ but A not the zero matrix.

Solution: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a possibility.

(b) (3 points) Matrices A and B such that $\det(A + B) \neq \det(A) + \det(B)$.

Solution: One example is as follows: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so $\det(A) = \det(B) = 0$ while $\det(A + B) = 1$.

(c) (4 points) Matrices A and B with $AB \neq A^T B^T$

Solution: We make things simpler by letting $A = I$ be the identity matrix, so then we desire $AB \neq A^T B^T$ which is the same as $B \neq B^T$. Thus any nonsymmetric matrix B will work.

7. (10 points) Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation such that:

$$T(at^2 + bt + c) = 2at + b - 2c$$

That is, $T(f(t)) = \frac{df}{dt} - 2f(0)$.

(a) (4 points) Write down a basis for the image of T.

> **Solution:** We see that the image doesn't contain any polynomials of degree 2. On the other hand, $T(t^2) = 2t$, and $T(t) = 1$, which are linearly independent. Thus, any linear polynomial is in the image, and therefore the image is precisely the subspace of linear polynomials. One basis is $B = \{2t, 1\}$.

(b) (2 points) What is the dimension of the image of T?

> **Solution:** It is the number of vectors in B which is 2.

(c) (4 points) Let $H \subset \mathbb{P}_2$ be the subset of polynomials of the form $2ct + c$, where c is any number. Use T to show that H is a subspace of $\mathbb{P}_2$.

> **Solution:** We see that H is actually the kernel of T, as if we have $T(at^2 + bt + c) = 0$, that means $2at + b - 2c = 0$, where 0 means the zero polynomial. Thus we must have $a = 0$ and $b = 2c$. That is, the kernel consists of the polynomials of the form $2ct + c$. Since H is the kernel of T, it is a subspace.