MATH 54

PROFESSOR KENNETH A. RIBET

Final Examination

December 17, 2005, 5–8 PM

Name and SID: GSI:

Please put away all books and electronic devices. You may refer to a single 2-sided sheet of notes. Your paper is your ambassador when it is graded. Correct answers without appropriate supporting work will be regarded skeptically. Incorrect answers without appropriate supporting work will receive no partial credit. This exam has 10 pages (and 9 problems). Please write your name on each page. At the conclusion of the exam, please hand in your paper to your GSI. The notations "**DE**" and "**FS**" are provided for Math 49 students. If you are one of those students, write "Math 49" prominently on the cover of your exam.

Problem	Your score	Total points	Math 49 info
1		4 points	
2		7 points	
3		8 points	
4		7 points	DE
5		7 points	FS
6		6 points	DE
7		8 points	
8		7 points	
9		6 points	FS
Total:		60 points	

1. Determine bases for the row and column spaces of the matrix $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 5 \\ -3 & -6 & 0 \end{bmatrix}$.

2. Let V be the vector space of 3×3 real matrices. Let W be the set of matrices $A \in V$ such that $A^{\mathrm{T}} = -A$. Is W a subspace of V? If so, find a basis for W.

Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

3. Let $v = [1, -1, -1, 1]^T$ and $w = [1, 1, 1, 1]$. For $x \in \mathbb{R}^4$, let $T(x) = (x \cdot v)v + (x \cdot w)w$, where "$\cdot$" is the usual dot product of vectors. Show that T is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$. Find two eigenvectors of T and one non-zero vector x such that $T(x) = 0$.

4. Let A be a 2×2 matrix such that

$$A\begin{bmatrix} -1 \\ 4 \end{bmatrix} = -4\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \quad A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -4\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

Find functions $x(t)$ and $y(t)$ with initial values $x(0) = -2$, $y(0) = 11$ that satisfy the system of differential equations $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

5. Suppose that $f(x) = 0$ for $-\pi < x < 0$, $f(x) = 1$ for $0 \leq x \leq \pi$, and $f(x+2\pi) = f(x)$ for $x \in \mathbb{R}$. As usual, write the Fourier series for $f(x)$ as $\frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$, Calculate the numbers a_m $(m \geq 0)$ and b_m $(m > 0)$.

6. Describe all pairs of numbers (y_0, y_0') such that the solution $y(t)$ to the initial value problem $y'' - 2y' - 3y = 0$, $y(0) = y_0$, $y'(0) = y_0'$ satisfies $y(t) \to 0$ as $t \to +\infty$.

7. Let A be a matrix whose null space is $\{0\}$. Explain carefully why each of the following statements is true:

(1) The rank of A equals the number of columns of A.
(2) The rows of A are linearly independent if and only if A is a square matrix;
(3) The product $A^T A$ of the transpose of A and A is an invertible matrix.

8. Let V be the vector space of all continuous functions on the real line. Consider the inner product $f \cdot g = \int_0^1 f(x)g(x)\,dx$ on V. Find a non-zero function that is orthogonal to the constant function 1 and to the functions x and x^2.

9. Solve the partial differential equation $100u_{xx} = u_t$ on the region $0 < x < 1$, $t > 0$, subject to the boundary conditions $u(0,t) = u(1,t)$ for $t > 0$ and $u(x,0) = \sin 2\pi x - \sin 5\pi x$ for $0 \leq x \leq 1$.