University of California, Berkeley
Department of Mathematics
12^{th} April, 2013, 12:10-12:55 pm

MATH 53 - Test #3

Last Name:________________Solutions______________________________

First Name:________________The_________________________________

Student Number:____

What is your discussion section number (201-215)? ____________________ [1]

What is the name of your GSI? __________________________________ [1]

Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.

Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.

There is a list of potentially useful formulas available on the last page of the exam.

For grader's use only:

Page	Grade
1	/2
2	/14
3	/12
4	/12
Total	/40

A

1. Evaluate the integral $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dy\,dx$. [6]

   (You can do it without reversing the order of integration, but it's not recommended.)

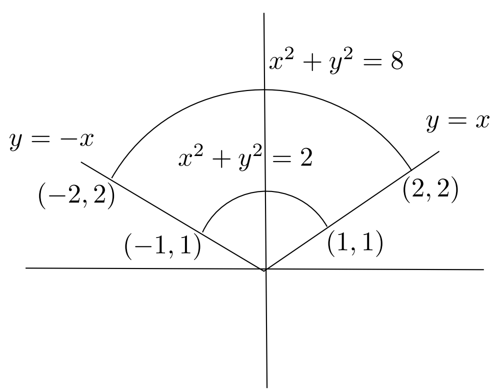
   The region of integration is the top half of the unit disk. As a Type II region it's given by $0 \le y \le 1$ with $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$, so

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dy\,dx &= \int_{0}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{1-y^2}\,dx\,dy \\ &= \int_0^1 2(1-y^2)\,dy \\ &= 2(1-1/3) = 4/3.\end{aligned}$$

2. The integral below computes the area of a region. Sketch the area, and compute it by converting to polar coordinates. [6]

$$\int_{-2}^{-1}\int_{-x}^{\sqrt{8-x^2}} dy\,dx + \int_{-1}^{1}\int_{\sqrt{2-x^2}}^{\sqrt{8-x^2}} dy\,dx + \int_{1}^{2}\int_{x}^{\sqrt{8-x^2}} dy\,dx$$

   The region given by the above integral lies above the x-axis between the circles $x^2+y^2=2$ and $x^2+y^2=8$ and the lines $y=\pm x$.



   In polar coordinates this region is given by $\pi/4 \le \theta \le 3\pi/4$ and $\sqrt{2} \le r \le 2\sqrt{2}$, so we have

$$\begin{aligned}A &= \int_{\pi/4}^{3\pi/4}\int_{\sqrt{2}}^{2\sqrt{2}} r\,dr\,d\theta \\ &= \frac{\pi}{2}\left(\frac{8-2}{2}\right) \\ &= \frac{3\pi}{2}.\end{aligned}$$

3. Evaluate the integral $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ by converting to polar coordinates. [6]

The half-circle $y = \sqrt{2x - x^2}$, or $x^2 + y^2 = 2x$ (which becomes $(x-1)^2 + y^2 = 1$ after completing the square) is given in polar coordinates by $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$. Since the region of integration is in the first quadrant, we have $0 \le \theta \le \pi/2$, so

$$\begin{aligned}
\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx &= \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\, dr\, d\theta \\
&= \frac{8}{3}\int_0^{\pi/2} \cos^3\theta\, d\theta \\
&= \frac{8}{3}\int_0^{\pi} 2(1-\sin^2\theta)\cos\theta\, d\theta \\
&= \frac{8}{3}\int_0^{1} (1-u^2)\, du \\
&= \frac{8}{3}\left(1 - \frac{1}{3}\right) \\
&= \frac{16}{9}.
\end{aligned}$$

4. Find the centroid (geometric center) of the triangle with vertices $(0,0)$, $(-4,2)$, and $(4,2)$. [8]

The triangle is given as a Type II region by $0 \le y \le 2$ with $-2y \le x \le 2y$. The triangle has base width 8 and height 2, so its area is $A = \frac{1}{2}(8)(2) = 8$. Since the region is symmetric about the y-axis we have

$$\overline{x} = \frac{1}{A}\iint_D x\, dA = 0$$

by symmetry, since $f(x,y) = x$ is an odd function of x. The y-coordinate of the centroid is given by

$$\begin{aligned}
\overline{y} &= \frac{1}{A}\iint_D y\, dA \\
&= \frac{1}{8}\int_0^2 \int_{-2y}^{2y} y\, dx\, dy \\
&= \frac{1}{8}\int_0^2 4y^2\, dy \\
&= \frac{1}{8}\left(\frac{4}{3}(2^3)\right) \\
&= \frac{4}{3}.
\end{aligned}$$

Thus, the centroid of the triangle is at $\left(0, \frac{4}{3}\right)$.

 Total Points: 40

5. Set up, but do not evaluate, the integral $\iiint_E x^2 \cos(yz)\, dV$, where E is the tetrahedron with vertices $(0,0,0)$, $(2,0,0)$, $(0,4,0)$, and $(0,0,1)$. [6]

The tetrahedron is bounded by the coordinate planes and the plane passing through the three points other than the origin. This plane intersects the coordinate planes in the lines $x+2z=2$, $y=0$ (or $2x+4z=4$), $y+4z=4$, $x=0$, and $2x+y=4$, $z=0$. Thus, the equation of the plane must be $2x+y+4z=4$, so we can describe the region by $0 \le y \le 4-2x-4z$ with (x,z) in the triangle bounded by $x=0$, $z=0$, and $x+2z=2$, which we can write as $0 \le x \le 2-2z$, with $0 \le z \le 1$. Thus, we have

$$\iiint_E x^2\cos(yz)\,dV = \int_0^1 \int_0^{2-2z} \int_0^{4-2x-4z} x^2 \cos(yz)\,dy\,dx\,dz.$$

6. Let $E \subseteq \mathbb{R}^3$ be the region bounded below by the cone $z=\sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2=2z$. Express the volume of E as a triple integral in **both** cylindrical and spherical coordinates. You do not have to compute the volume. [6]

The sphere $x^2+y^2+z^2=2z$, or $x^2+y^2+(z-1)^2=1$, intersects the cone $z=\sqrt{x^2+y^2}$ when $z^2+z^2=2z$, which gives $z=0$ or $z=1$. The intersection at $z=0$ is where the base of the cone meets the bottom of the sphere, while the intersection at $z=1$ is the circle $z=1=x^2+y^2$. The region thus consists of the top half of the sphere, lying on top of the portion of the cone between $z=0$ and $z=1$.

In cylindrical coordinates, we have $0 \le \theta \le 2\pi$ and $0 \le r \le 1$, since the region projects down to the xy-plane onto the disk $x^2+y^2 \le 1$. The equation of the cone is simply $z=r$, while the sphere becomes $(z-1)^2 = 1-r^2$, so $z = 1+\sqrt{1-r^2}$ for the top half of the sphere (positive root). Thus, the volume is given in cylindrical coordinates by

$$V = \int_0^{2\pi}\int_0^1\int_r^{1+\sqrt{1-r^2}} r\,dz\,dr\,d\theta.$$

In spherical coordinates the cone is given by $\varphi=\pi/4$, so the interior of the cone corresponds to $0\le\varphi\le\pi/4$, and the sphere is given by $\rho^2 = 2\rho\cos\varphi$, or $\rho = 2\cos\varphi$ (since $\rho=0$ can be obtained by setting $\varphi=\pi/2$). Since the region lies inside the sphere, we have $0\le\rho\le 2\cos\varphi$, and since the region is symmetric about the z-axis we have $0\le\theta\le 2\pi$. Thus, we have

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

 Total Points: 40