*University of California, Berkeley*
Department of Mathematics
$15^{\text{th}}$ March, 2013, 12:10-12:55 pm

**MATH 53 - Test #2**

Last Name: Solutions

First Name: The

Student Number: ______

Discussion Section: ______

Name of GSI: ______

**Record your answers below each question in the space provided. Left-hand pages may be used as scrap paper for rough work. If you want any work on the left-hand pages to be graded, please indicate so on the right-hand page.**

**Partial credit will be awarded for partially correct work, so be sure to show your work, and include all necessary justifications needed to support your arguments.**

**There is a list of potentially useful formulas available on the last page of the exam.**

For grader's use only:

| Page | Grade |
|---|---|
| 1 | /15 |
| 2 | /15 |
| 3 | /10 |
| Total | /40 |

A

1. Let $f(x, y) = x^2 e^{xy}$.

[4] (a) Find the linearization of $f$ at the point $(1, 0)$.

The partial derivatives of $f$ are $f_x(x, y) = 2xe^{xy} + x^2 ye^{xy}$ and $f_y(x, y) = x^3 e^{xy}$, so the linearization of $f$ at $(1, 0)$ is given by

$$\begin{aligned} L(x, y) &= f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) \\ &= 1 + 2(x - 1) + y = 2x + y - 1. \end{aligned}$$

[3] (b) Find the derivative of $f$ in the direction of $\mathbf{v} = \langle -3, 4\rangle$ at the point $(1, 0)$.

The directional derivative is given by

$$D_{\mathbf{v}} f(1, 0) = \frac{\nabla f(1, 0) \cdot \mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 2, 1\rangle \cdot \langle -3, 4\rangle}{\sqrt{(-3)^2 + 4^2}} = \frac{-2}{5}.$$

[5] (c) If $x(t) = t^2$ and $y(t) = 2t - 2$, use the chain rule to find the tangent vector to the curve $\mathbf{r}(t) = \langle x(t), y(t), f(x(t), y(t))\rangle$ when $t = 1$.

We have $x'(t) = 2t$, $y'(t) = 2$ and

$$z'(t) = \frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

When $t = 1$, $x(1) = 1$, $y(1) = 0$, $x'(1) = 2$, and $y'(1) = 2$. Thus, $z'(1) = f_x(1, 0)x'(1) + f_y(1, 0)y'(1) = 2(2) + 1(2) = 6$, so $\mathbf{r}'(1) = \langle 2, 2, 6\rangle$.

[3] (d) Verify that the tangent vector found in part (c) is tangent to the surface $z = f(x, y)$ at the point $(1, 0, 1)$.

The tangent plane is given by $z = L(x, y) = 2x + y - 1$, so a normal vector is $\mathbf{n} = \langle 2, 1, -1\rangle$. Since $\mathbf{n} \cdot \mathbf{r}'(1) = \langle 2, 1, -1\rangle \cdot \langle 2, 2, 6\rangle = 4 + 2 - 6 = 0$, $\mathbf{r}'(1)$ must lie in the tangent plane at $(1, 0, 1)$.

2. Let $f(x, y) = 3xy - x^3 - y^3$.

[8] (a) Find and classify the critical points of $f$.

The gradient of $f$ is given by

$$\nabla f(x, y) = \langle 3y - 3x^2, 3x - 3y^2 \rangle,$$

which vanishes when $y = x^2$ and $x = y^2$. Plugging the first equation into the second gives $x = x^4$, so $x = 0$ or $x = 1$. This gives us the critical points $(0, 0)$ and $(1, 1)$.

The second derivatives of $f$ are given by $f_{xx}(x, y) = -6x$, $f_{xy}(x, y) = 3$, and $f_{yy}(x, y) = -6y$. Thus, we have

- At $(0, 0)$, $A = 0$, $B = 3$, $C = 0$, and $D = AC - B^2 = -9 < 0$, so $(0, 0)$ is a saddle point.
- At $(1, 1)$, $A = -6 < 0$, $B = 3$, $C = -6$, and $D = AC - B^2 = 27 > 0$, so $(1, 1)$ is a local maximum.

[7] (b) Find the absolute maximum and minimum of $f$ on the set $D$ given by the triangular region with vertices at $(0, 0)$, $(0, 1)$, and $(1, 1)$.

From part (a) we have the critical values $f(0, 0) = 0$ and $f(1, 1) = 1$, which occur at points in $D$, since $(0, 0)$ and $(1, 1)$ are two of the three vertices of the triangle. The value of $f$ at the remaining vertex is $f(0, 1) = -1$.

Since there are no critical points on the interior of $D$, it remains to check for any boundary extrema. The boundary consists of three lines: $x = 0$, $y = 1$, and $x = y$. Since we've already checked the end points of these lines (the vertices of the triangle), we just have to check for critical points of the restriction of $f$ to each line.

For $x = 0$ we get $f(0, y) = -y^3$, which has its only critical point when $y = 0$ which gives the point $(0, 0)$, which we've already checked. For $y = 1$ we get $g(x) = f(x, 1) = 3x - x^3 + 1$. We have $g'(x) = 3 - 3x^2$, so $g$ has critical points when $x = \pm 1$, but $(-1, 1)$ is not in $D$, and $(1, 1)$ we've already checked.

Finally, if $x = y$ we have $h(x) = f(x, x) = 3x^2 - 2x^3$, so $h'(x) = 6x - 6x^2$, and $h$ has critical points $x = 0$ and $x = 1$, corresponding to the points $(0, 0)$ and $(1, 1)$, which we've already checked. Having exhausted all possibilities, we conclude that the absolute maximum is $f(1, 1) = 1$, and the absolute minimum is $f(0, 1) = -1$.

3. (a) Define what it means for a function $f(x, y, z)$ to be *continuous* at a point $(a, b, c)$ in

[2] its domain.

A function $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is continuous at $(a, b, c) \in D$ if

$$\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = f(a, b, c).$$

(b) Define what it means for a function $f(x, y, z)$ to be *differentiable* at a point $(a, b, c)$ in

[3] its domain.

A function $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is differentiable at $(a, b, c) \in D$ if

$$\lim_{(x,y,z)\to(a,b,c)} \frac{f(x, y, z) - f(a, b, c) - \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c\rangle}{\sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}} = 0.$$

[5] (c) Show that if $f$ is differentiable at a point $(a, b, c)$, then it is continuous at $(a, b, c)$. *Hint:* You can show this using only the above two definitions and the limit laws.

Let $\mathbf{x} = \langle x, y, z\rangle$ be the position vector for $(x, y, z)$, and similarly define $\mathbf{a} = \langle a, b, c\rangle$. Suppose that $f$ is differentiable at $\mathbf{a}$. Then we have

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) &= \lim_{\mathbf{x}\to\mathbf{a}} \left(f(\mathbf{x}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + f(\mathbf{a})\right) \\
&= \lim_{\mathbf{x}\to\mathbf{a}} \left[\left(\frac{f(\mathbf{x}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})}{\|\mathbf{x} - \mathbf{a}\|}\right)(\|\mathbf{x} - \mathbf{a}\|)\right] \\
&\qquad + \lim_{\mathbf{x}\to\mathbf{a}} \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{a}) \\
&= 0(0) + \nabla f(\mathbf{a}) \cdot \mathbf{0} + f(\mathbf{a}) \\
&= f(\mathbf{a}).
\end{aligned}$$

Thus, $f$ is continuous at $\mathbf{a}$.

 **Total Points: 40**