Mathematics 185. Fall Semester 2014 **Instructor: Mihaela Ifrim**

Midterm

October 15, 2014, 15:10-16:00

Your Name: ________________________________

Your ID: ________________________________

Directions: This is a *closed* book exam. No calculators, cell phones, pagers, mp3 players and other electronic devices are allowed.
Remember: Answers without explanations will not count. You should **show your work**. Solve each problem on its own page. If you need extra space you can use backs of the pages and the extra page attached to your exam paper, but make a note you did so.

Problem	Score
1	
2	
3	
4	
5	
Total	
Grade	

(20) 1 . Show that

$$\int_{-\infty}^{+\infty} \frac{x}{(x^2+2x+2)(x^2+4)}\,dx = -\frac{\pi}{10}.$$

Proof. This integral fits in the $\int \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x, and the degree of $P(x)$ is less with more than 2 units than the degree of $Q(x)$. Thus the contour we will work with is the semicircle countour; depending on your preference we can work with the uper half one or with the lower one. Assuming you prefere the upper one, we need to figure out which are the solutions of $Q(z) = 0$ that are enclosed by this contour. We get $z = 2i, -2i, -1-i, -1+i$. Thus the poles that live in the upper half plane and can be anclosed by a semisicre of radius R, are $z = 2i$ and $z = -1+i$. Thus the whole contour we are integrating over, C_R, is the union of the real segment $[-R, R]$ and the upper bound of the semicircle, which we will denote by Γ_R; this means $C_R = [-R, R] \cup \Gamma_R$.
Integrating over the countour we have

$$\int_{C_R} \frac{z}{(z^2+2z+2)(z^2+4)}\,dz = \int_{\Gamma_R} \frac{z}{(z^2+2z+2)(z^2+4)}\,dz + \int_{-R}^{R} \frac{x}{(x^2+2x+2)(x^2+4)}\,dx. \quad (1)$$

By Residue Formula the RHS of(1) is $2\pi i\times$ residue of f at $2i$ added witht the $2\pi i\times$residue of f at $-1+i$, were

$$f = \frac{z}{(z^2+2z+2)(z^2+4)}.$$

Easy computations gives

$$2\pi i \mathrm{res}_{2i} f = \frac{-\pi i - 2\pi}{10}, \quad 2\pi i \mathrm{res}_{-1+i} f = \frac{\pi i - 3\pi}{10}.$$

$$\frac{-\pi i - 2\pi}{10} + \frac{\pi i - 3\pi}{10} = -\frac{\pi}{10} = \int_{\Gamma_R} \frac{z}{(z^2+2z+2)(z^2+4)}\,dz + \int_{-R}^{R} \frac{x}{(x^2+2x+2)(x^2+4)}\,dx.$$

□

Making R go to infinity we observe that

$$\int_{\Gamma_R} \frac{z}{(z^2+2z+2)(z^2+4)}\,dz == \int_0^{\pi} \frac{iR^2e^{2i\theta}}{(R^2e^{2i\theta}+2Re^{\imath\theta}+2)(R^2e^{2i\theta}+4)}\,d\theta$$

Now, (use L'Hospital Rule or realize that the power of the exponential il much larger in the denominator and it alsways "win")

$$\lim_{R\to\infty} \frac{iR^2e^{2i\theta}}{(R^2e^{2i\theta}+2Re^{\imath\theta}+2)(R^2e^{2i\theta}+4)} = 0.$$

Thus, as $R \to \infty$

$$-\frac{\pi}{10} = \int_{-\infty}^{\infty} \frac{x}{(x^2+2x+2)(x^2+4)}\,dx.$$

This finishes the proof.

(20) 2. Please state the following

- (a) Cauchy's Integral Formula.

- (b) Maximum Modulus Principle.

(20) 3. Use Rouche's Theorem to find the number of zeros of the polynomial z^4+5z+3 in the annulus $1<|z|<2$.

Proof. First we count the zeros for $|z|<2$. We pick $f(z)=z^4$ and observe that on $|z|=2$, $|f(z)|=16$. The remaning part of the polynomial will be called $g(z):=5z+3$. And on $|z|=2$, $|g(z)|\leq 5|z|+3=13$. Thus, f has the same numbers of zero as $f+g$. Since f has four zeros inside the circle of radius 2, we conclude that so has our polynomial.

Secondly we count the zeros for $|z|<1$. We pick $f(z)=5z+3$ and observe that on $|z|=1$, $2\leq|f(z)|\leq 8$. The remaning part of the polynomial will be called $g(z):=z^4$. And on $|z|=1$, $|g(z)|=1$. Thus, f has the same numbers of zero as $f+g$. Since f has one zero inside the unit circle (the zero is attained at $z=-3/5$, which is in the unit disc), we conclude that so has our polynomial.

Thus, one can conclude that in the annulus $1<|z|<2$ our polynomial should have three solutions... But this is not quite that obvious. We only obtained that there are three zeros in $1\leq|z|<2$. What if some of the solutions enclosed inside our circle of radius 2 sit exactly on the boundary of the unit circle (note that we have strict inequality in the statement of the problem)? To make the first inequality strict, we also need to show that our polynomis has no zeros on $|z|=1$. Indeed, in this case

$$|5z+3-(-z^4)|\geq ||5z+3|-|-z^4||=||5z+3|-1|.$$

And just above we showed above that $|5z+3|>2$ on $|z|=1$. Thus, $|f(z)|>1$ on $|z|=1$. $\square$

(20) 4 . Suppose that $f(z)$ is an entire function such that

$$|f(z)| \leq e^{\Re z}, \quad \text{for all } z \in \mathbf{C}.$$

Prove that $f(z) = ce^z$ for some constant $c \in \mathbf{C}$.

Proof. The function$f(z)/e^z$ is entire and bounded (since e^z is entire and non-zero, and $|f(z)| \leq |e^z|$), so by Liouvilles theorem it is equal to a constant. $\square$

(20) 5. Let f be non-constant and holomorphic in an open set containing the closed unit disc. Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of f contains the unit disc.
Hint: One must show that $f(z) = w_0$ has a root for every $w_0 \in D$. To do this, it suffices to show that $f(z) = 0$ has a root (why?). Use the maximum modulus principle to conclude.

Proof. Assume $|f(z)| = 1$ whenever $|z| = 1$, and let w_0 in D. Applying Rouche's theorem to the functions $f(z)$, and $g(z) = -w_0$, we see that

$$|f(z)| = 1 > |-w_0| = |g(z)|$$

for all z on ∂D (this notation meas boundary of D), and therefore $f(z)$ and $f(z)+g(z) = f(z)-w_0$ have the same number of zeros in D. Hence, if $f(z) = 0$ has a solution, then $f(z) = w_0$ has a solution.
We are reduced to showing $f(z) = 0$ for some $z \in D$. Assume not. Then the function $1/f(z)$ is holomorphic on D , and the maximum modulus principle implies

$$\frac{1}{\inf_{z\in D}(|f(z)|)} = \sup_{z\in D}\left(\frac{1}{|f(z)|}\right) \leq \sup_{|z|=1}\left(\frac{1}{(|f(z)|}\right) = 1.$$

Applying the maximum modulus principle to $f(z)$, we obtain

$$\sup_{z\in D}(|f(z)|) \leq \sup_{z\in\partial D}(|f(z)|) = 1.$$

Hence, if $z \in D$, we have

$$1 \leq \inf_{z\in D}(|f(z)|) \leq |f(z)| \leq \sup_{z\in D}(|f(z)|) \leq 1,$$

which implies $f(z)$ is constant by Chapter 1, Problem 13. We thus have a contradiction, and therefore the image of f contains D. $\square$