Math 110, Spring 2015: Midterm Solutions

These are not intended as "model answers"; in many cases far more explanation is provided than would be necessary to receive full credit. The goal here is to make sure you understand the questions and their solutions. Accordingly, don't feel obligated to read all of this in detail; if you were confused or curious about a particular question, you can read the discussion to drill down into it. **That said, it is far more productive to try to work out the correct solutions for yourself first.**

Question 1. *For each statement below, determine whether it is true or not.*

(1) *The space of degree n polynomials in one variable is a vector space.*

Solution: False. The space in question is not closed under addition, as the sum of two degree-n polynomials may have degree strictly less than n. (The space of polynomials in one variable of degree *less than or equal to* n is a vector space over the coefficient field F; for $F = \mathbb{R}$, this space is the familiar $\mathrm{P}_n(\mathbb{R})$ you have worked with throughout the course.)

(2) *The dimension of the vector space of $n \times n$ matrices such that $A^t = A$ is equal to $n(n-1)/2$.*

Solution: False. The correct dimension is $\frac{n^2-n}{2} + n = \frac{n(n+1)}{2}$. (Roughly speaking, an incorrect answer of "True" might result from "failing to account for the diagonals" of symmetric $n \times n$ matrices.)

(3) *Any subset of a set of linearly dependent vectors is linearly dependent.*

Solution: False. For example, to take this statement to its extreme, consider the set V of *all* vectors in the vector space V; this set is linearly dependent, as it contains the zero vector of V. So if the statement were true, it would mean that *any subset of V is linearly dependent*; in other words, there would be *no such thing as a "linearly independent set"* in any vector space. Hopefully that example makes the statement seem absurd.

(4) *Any two vector spaces of dimension n are isomorphic to each other.*

Solution: True. This is proven in the book. One particularly important way of looking at this is to note that—if one *chooses a basis* β of an n-dimensional vector space V—the "coordinate mapping" $x \mapsto [x]_\beta$ gives an isomorphism from V to $\mathbb{R}^n$. More generally, say V and W are n-dimensional vector spaces with bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$, respectively. Then there exists a unique linear transformation $\mathrm{T} : V \to W$ such that $\mathrm{T}(v_k) = w_k$ for all $1 \leq k \leq n$ (by Theorem 2.6 in the book); by a homework exercise, T is an isomorphism. **NB:** It has been emphasized to you that, so far in this course, all vector spaces are assumed to be over the field $\mathbb{R}$. So in the context of the course thus far, considering two vector spaces of dimension n over two *different fields* would miss the point of this question.

(5) *If a set S of vectors generates a vector space V, then any vector in V can be written uniquely as a linear combination of the vectors in S.*

Solution: False. The problem is with the word "uniquely"; the statement fails for any generating set S that is not also *linearly independent*, *i.e.* for any generating set S that is not a *basis* for V.

(6) *The intersection of any two subspaces of a vector space V is a subspace of V.*

Solution: True. Proving this is a basic application of the definition of "subspace."

(7) *If V is isomorphic to W, then V^* is isomorphic to W^*.*

Solution: True. I believe this was proven in lecture.

Remark: Here is a line of reasoning that works in the *finite-dimensional* setting: "By the construction of the dual basis, if V is finite-dimensional, then $\dim(V) = \dim(V^*)$, so V is isomorphic to V^*. Similarly, W is isomorphic to W^*. Thus, because isomorphism of vector spaces is an equivalence relation, V^* must be isomorphic to W^*." *That line of reasoning fails if V* (and hence W) *is infinite-dimensional*; indeed, an infinite-dimensional vector space is *never* isomorphic to its dual space.[1] So the best way to reason through this question is to prove the following result, of which the question is the special case $V' = W' = \mathbb{R}$:

Proposition 1. *Suppose V is isomorphic to W and V' is isomorphic to W'. Then $\mathcal{L}(V, V')$ is isomorphic to $\mathcal{L}(W, W')$.*

Proof. Let $\varphi : V \to W$ and $\varphi' : V' \to W'$ be isomorphisms. Then check that the map $\Phi : \mathcal{L}(V, V') \to \mathcal{L}(W, W')$ defined by $\Phi(\mathrm{T}) = \varphi' \circ \mathrm{T} \circ \varphi^{-1}$ is an isomorphism. (Draw a "commutative diagram.") □

(8) *If V and W are two vector spaces of the same dimension and $\mathrm{T} : V \to W$ is a linear transformation, then there exist bases in V and W such that the matrix of T with respect to these bases is diagonal.*

Solution: True. This was a homework exercise (Exercise 2.2.16 of Homework 4).

(9) *If A and B are two matrices such that AB is the identity matrix, then BA is also the identity matrix.*

Solution: False. The statement is true *if one assumes additionally that A and B are* square *matrices.* (See Exercise 2.4.10.) A counterexample to the general statement given here is

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(10) *If $\mathrm{T} : V \to V$ is a linear transformation and A_1, A_2 are the matrices representing T with respect to bases β_1, β_2 in V, then the trace of A_1 is equal to the trace of A_2.*

Solution: True. Let $Q = [\mathrm{I}]_{\beta_1}^{\beta_2}$ be the change-of-coordinates matrix that changes β_1-coordinates into β_2-coordinates. Then we have (see Theorem 2.23 and its proof)

$$A_1 = [\mathrm{T}]_{\beta_1}^{\beta_1} = [\mathrm{I} \circ \mathrm{T} \circ \mathrm{I}]_{\beta_1}^{\beta_1} = [\mathrm{I}]_{\beta_2}^{\beta_1} [\mathrm{T}]_{\beta_2}^{\beta_2} [\mathrm{I}]_{\beta_1}^{\beta_2} = Q^{-1} A_2 Q.$$

In particular, A_1 and A_2 are *similar* matrices; by Exercise 2.5.10 of Homework 5, $\mathrm{tr}(A_1) = \mathrm{tr}(A_2)$.

[1] Purely extracurricular comment: I would not want this statement to confuse you if you encounter "dual spaces" in analysis courses. Analysts use a different notion of the "continuous dual" of a vector space, and there are infinite-dimensional vector spaces that are isomorphic to their continuous dual spaces in an appropriate sense. By contrast, in this course we consider the "algebraic dual" of a vector space V, and no infinite-dimensional vector space V is isomorphic to its algebraic dual space.

Question 2: *Determine whether the following polynomials are linearly dependent or independent in the vector space* $\mathrm{P}_2(\mathbb{R})$ *of real polynomials of degree less than or equal to* 2.

(a) $1 + x + x^2$, $-1 + 2x^2$, $3 + x + 4x^2$;

Solution: These three polynomials are linearly independent. Before starting to work, one should note that $\dim\big(\mathrm{P}_2(\mathbb{R})\big) = 3$, so it is *possible* for a set of three vectors to be linearly independent. Accordingly, we just check the definition of linear independence: We want to check whether or not a "dependence relation"

$$c_1(1 + x + x^2) + c_2(-1 + 2x^2) + c_3(3 + x + 4x^2) = \mathit{0}_{\mathrm{P}_2(\mathbb{R})}$$

forces the coefficients c_1, c_2, c_3 all to be zero. The dependence relation is equivalent to the system of linear equations

$$\begin{cases} c_1 - c_2 + 3c_3 = 0 \\ c_1 + c_3 = 0 \\ c_1 + 2c_2 + 4c_3 = 0. \end{cases}$$

Solving the system yields the unique solution $c_1 = c_2 = c_3 = 0$, so the three polynomials are indeed linearly dependent.

(c) $13 + 2x + x^2$, $2 + 8x + 2x^2$, $x + 26x^2$, $15 + 6x + 9x^2$.

Solution: These polynomials are linearly dependent. Indeed, since $4 > 3 = \dim\big(\mathrm{P}_2(\mathbb{R})\big)$, any set of four distinct polynomials in $\mathrm{P}_2(\mathbb{R})$ must be linearly dependent.

Remark: Even if you went through the trouble of setting up a system of linear equations as in part (a), you could notice that it would be a homogeneous linear system of four equations in three unknowns; hopefully you learned in Math 54 that such a system must have infinitely many solutions. That's just a long-winded way of saying "four distinct vectors in a three-dimensional vector space must be linearly dependent."

Question 3: *Let* $\mathrm{M}^0_{3\times 3}(\mathbb{R})$ *be the subset of the vector space* $\mathrm{M}_{3\times 3}(\mathbb{R})$ *of all real* 3×3 *matrices, which consists of matrices* A *whose column sums are* 0*, that is*

$$A_{1i} + A_{2i} + A_{3i} = 0; \quad i = 1, 2, 3.$$

(a) *Prove that* $\mathrm{M}^0_{3\times 3}(\mathbb{R})$ *is a subspace of* $\mathrm{M}_{3\times 3}$*;*

Solution: First we note that the zero matrix $\mathit{0} \in \mathrm{M}_{3\times 3}(\mathbb{R})$ is an element of $\mathrm{M}^0_{3\times 3}(\mathbb{R})$. Indeed, since all entries of $\mathit{0}$ are zero, certainly all of its columns must sum to zero. To conclude, we simply verify that $\mathrm{M}^0_{3\times 3}(\mathbb{R})$ is closed under addition and scalar multiplication. Let $A, B \in \mathrm{M}^0_{3\times 3}(\mathbb{R})$ and $c \in \mathbb{R}$ be arbitrary. Then, for each $i = 1, 2, 3$, we have

$$(A + cB)_{1i} + (A + cB)_{2i} + (A + cB)_{3i} = (A_{1i} + A_{2i} + A_{3i}) + c(B_{1i} + B_{2i} + B_{3i}) = 0 + c \cdot 0 = 0,$$

so $A + cB \in \mathrm{M}^0_{3\times 3}(\mathbb{R})$. Thus, $\mathrm{M}^0_{3\times 3}(\mathbb{R})$ is closed under addition and scalar multiplication.

(b) *Construct a basis of* $\mathrm{M}^0_{3\times 3}$.

Solution: To find a basis, it's certainly very helpful to know the dimension of $\mathrm{M}^0_{3\times 3}(\mathbb{R})$. To compute the dimension, note that $\mathrm{M}^0_{3\times 3}(\mathbb{R})$ can naturally be expressed as the kernel of a linear transformation. Namely, consider $\mathrm{T} : \mathrm{M}_{3\times 3}(\mathbb{R}) \longrightarrow \mathbb{R}^3$ defined by

$$\mathrm{T}(A) = (A_{11} + A_{21} + A_{31}\ ,\ A_{12} + A_{22} + A_{32}\ ,\ A_{13} + A_{23} + A_{33});$$

that is, the j-th entry of $\mathrm{T}(A)$ is the j-th column sum of A. Then *by definition* $\mathrm{M}^0_{3\times 3}(\mathbb{R}) = \mathrm{N}(\mathrm{T})$. Furthermore, it's pretty easy to see that T is onto; I'll leave that as an exercise. So by the Dimension Theorem,

$$\mathrm{rank}(\mathrm{T}) + \mathrm{nullity}(\mathrm{T}) = \dim\left(\mathbb{R}^3\right) + \dim\left(\mathrm{M}^0_{3\times 3}(\mathbb{R})\right) = \dim\left(\mathrm{M}_{3\times 3}(\mathbb{R})\right) = 9,$$

so $\mathrm{M}^0_{3\times 3}(\mathbb{R})$ has dimension 6. So any collection of six linearly independent matrices in $\mathrm{M}^0_{3\times 3}(\mathbb{R})$ must form a basis for the subspace; a natural choice of such a collection is

$$S = \{E_{ij} - E_{3j}\,|\, i = 1, 2;\ j = 1, 2, 3\},$$

where E_{ij} is the matrix whose (i, j) entry is 1 and all of whose other entries are 0. Showing that S is linearly independent is fairly straightforward.

Question 4: *Consider a function* $\mathrm{T}: \mathrm{P}_3(\mathbb{R}) \to \mathrm{P}_2(\mathbb{R})$*, such that* $\mathrm{T}\big(p(x)\big) = p'(x) + 5p''(x)$*. (Here* $\mathrm{P}_n(\mathbb{R})$ *denotes the vector space of real polynomials of degree less than or equal to* n *in one variable.)*

(a) *Prove that* T *is a linear transformation;*

Solution: Let $p(x), q(x) \in \mathrm{P}_3(\mathbb{R})$ and $\lambda \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned} \mathrm{T}\big(\lambda p(x) + q(x)\big) &= (\lambda p + q)'(x) + 5(\lambda p + q)''(x) \\ &= (\lambda p)'(x) + q'(x) + 5(\lambda p)''(x) + 5q''(x) \\ &= \lambda p'(x) + q'(x) + 5\lambda p''(x) + 5q''(x) \\ &= \lambda\big(p'(x) + 5p''(x)\big) + \big(q'(x) + q''(x)\big) \\ &= \lambda \mathrm{T}\big(p(x)\big) + \mathrm{T}\big(q(x)\big), \end{aligned}$$

where we've used linearity of the differentiation operator. So T is linear.

(b) *Choose bases* β *of* $\mathrm{P}_3(\mathbb{R})$ *and* γ *of* $\mathrm{P}_2(\mathbb{R})$ *and find the matrix* $[\mathrm{T}]_\beta^\gamma$ *of* T *with respect to these bases.*

Solution: By far the most popular choice of bases was $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2\}$; these bases make the computations involved particularly simple. The j-th column of $[\mathrm{T}]_\beta^\gamma$ is the γ-coordinate vector of the image under T of the j-th basis vector in β. So we just compute those coordinate vectors:

$$\mathrm{T}(1) = (1)' + 5(1)'' = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \implies [\mathrm{T}(1)]_\gamma = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

$$\mathrm{T}(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \implies [\mathrm{T}(x)]_\gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};$$

$$\mathrm{T}(x^2) = 2x + 10 = 10 \cdot 1 + 2 \cdot x + 0 \cdot x^2 \implies [\mathrm{T}(x^2)]_\gamma = \begin{bmatrix} 10 \\ 2 \\ 0 \end{bmatrix};$$

$$\mathrm{T}(x^3) = 3x^2 + 30x = 0 \cdot 1 + 30 \cdot x + 3 \cdot x^2 \implies [\mathrm{T}(x^3)]_\gamma = \begin{bmatrix} 0 \\ 30 \\ 3 \end{bmatrix}.$$

So

$$[\mathrm{T}]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 10 & 0 \\ 0 & 0 & 2 & 30 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Question 5:

(a) *Give the definition of an isomorphism between two vector spaces.*

Solution 1: "An isomorphism between two vector spaces V and W is an invertible linear transformation $\mathrm{T} : V \to W$."

Solution 2: "An isomorphism between two vector spaces V and W is a map $\mathrm{T} : V \to W$ that is linear, one-to-one (injective), and onto (surjective)." (Of course, the two definitions are equivalent.)

(b) *Prove that if there exists an isomorphism between two finite-dimensional vector spaces, then these vector spaces have the same dimension.*

Solution 1: Suppose $\mathrm{T} : V \to W$ is an isomorphism between the finite-dimensional vector spaces V and W. Then because T is one-to-one, its kernel is trivial, *i.e.* $\mathrm{N(T)} = \{\mathit{0}\} \subseteq V$. Furthermore, because T is onto, its range must be all of W, *i.e.* $\mathrm{R(T)} = W$. By the Dimension Theorem,

$$\dim(V) = \dim\big(\,\mathrm{N(T)}\,\big) + \dim\big(\,\mathrm{R(T)}\,\big) = \dim\big(\{\mathit{0}\}\big) + \dim(W) = 0 + \dim(W) = \dim(W),$$

as desired.

Solution 2: Suppose $\mathrm{T} : V \to W$ is an isomorphism between the finite-dimensional vector spaces V and W. Fix a basis $\beta = \{v_1, \ldots, v_n\}$, and consider the set

$$\gamma := \{\mathrm{T}(v_1), \ldots, \mathrm{T}(v_n)\} \subseteq W;$$

we claim γ is a basis for W. To see that γ is linearly independent, suppose that $c_1\mathrm{T}(v_1) + \ldots + c_n\mathrm{T}(v_n) = \mathit{0}_W$ for some scalars c_k. But then

$$\mathrm{T}(c_1v_1 + \ldots + c_nv_n) = \mathit{0} \Longrightarrow c_1v_1 + \ldots + c_nv_n \in \mathrm{N(T)} \Longrightarrow c_1v_1 + \ldots + c_nv_n = \mathit{0}_V$$

since T is one-to-one. But since β is a basis, the v_k are linearly independent, and so the c_k must all be equal to zero. So γ is linearly independent.

To see that γ spans W, let $w \in W$ be arbitrary. Because T is onto, $w \in \mathrm{R(T)}$, so there exists some $v \in V$ such that $\mathrm{T}(v) = w$. Writing v in terms of the basis β, by linearity of T we obtain

$$w = \mathrm{T}(v) = \mathrm{T}(a_1v_1 + \ldots + a_nv_n) = a_1\mathrm{T}(v_1) + \ldots + a_n\mathrm{T}(v_n)$$

for some scalars a_k, so that $w \in \mathrm{span}(\gamma)$. Since w was arbitrary, we've shown that γ spans W. So we see that γ is a basis for W; since $\gamma = \{\mathrm{T}(v_1), \ldots, \mathrm{T}(v_n)\}$ contains n vectors, we see that

$$\dim(W) = n = \dim(V).$$

Question 6: *Let V be a two-dimensional vector space and $\mathrm{T} : V \to V$ a linear transformation. Suppose that $\beta = \{x_1, x_2\}$ and $\gamma = \{y_1, y_2\}$ are two bases in V such that*

$$x_1 = y_1 + y_2, \quad x_2 = y_1 + 2y_2.$$

Find $[\mathrm{T}]_\gamma^\gamma$ if

$$[\mathrm{T}]_\beta^\beta = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}.$$

Solution 1 (direct computation): This approach just uses the definition of the matrix representation of a linear transformation with respect to a pair of bases. From that definition, we know that the j-th column of $[\mathrm{T}]_\beta^\beta$ is the β-coordinate vector of $\mathrm{T}(x_j)$. So we have

$$[\mathrm{T}(x_1)]_\beta = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \implies \mathrm{T}(x_1) = 2 \cdot x_1 + 1 \cdot x_2 = 2(y_1 + y_2) + 1(y_1 + 2y_2) = 3y_1 + 4y_2;$$

$$[\mathrm{T}(x_2)]_\beta = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \implies \mathrm{T}(x_2) = 3x_1 + 4x_2 = 7y_1 + 11y_2.$$

(Here we've used the given expressions for x_j in terms of the y_k.) Since we want $[\mathrm{T}]_\gamma^\gamma$, we need to compute $\mathrm{T}(y_1)$ and $\mathrm{T}(y_2)$. From what we've already done, we could compute these if we knew how to express y_1 and y_2 as linear combinations of the x_k. From the equations $x_1 = y_1 + y_2$ and $x_2 = y_1 + 2y_2$, it's pretty easy to solve for the y_j: $y_1 = 2x_1 - x_2$, $y_2 = -x_1 + x_2$. So we compute

$$\mathrm{T}(y_1) = \mathrm{T}(2x_1 - x_2) = 2\mathrm{T}(x_1) - \mathrm{T}(x_2) = -y_1 - 3y_2 \implies [\mathrm{T}(y_1)]_\gamma = \begin{bmatrix} -1 \\ -3 \end{bmatrix};$$

$$\mathrm{T}(y_2) = \mathrm{T}(-x_1 + x_2) = 4y_1 + 7y_2 \implies [\mathrm{T}(y_2)]_\gamma = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

So $[\mathrm{T}]_\gamma^\gamma = \begin{bmatrix} -1 & 4 \\ -3 & 7 \end{bmatrix}$.

Solution 2(change-of-coordinates matrix): Remark: Many people took this approach, and many people wrote down the matrix that changes *γ-coordinates into β-coordinates* instead of vice versa. Carrying this out carefully step-by-step as here might help you avoid that error.

As in the discussion of part (10) of Question 1 above, we have

$$[\mathrm{T}]_\gamma^\gamma = [\mathrm{I}]_\beta^\gamma \, [\mathrm{T}]_\beta^\beta \, [\mathrm{I}]_\gamma^\beta.$$

We can compute the *inverse* of the change-of-coordinates matrix $Q := [\mathrm{I}]_\gamma^\beta$ as follows:

$$Q^{-1} = [\mathrm{I}]_\beta^\gamma = \Big[[\mathrm{I}(x_1)]_\gamma \, [\mathrm{I}(x_2)]_\gamma \Big] = \Big[[x_1]_\gamma \, [x_2]_\gamma \Big] = \Big[[y_1 + y_2]_\gamma \, [y_1 + 2y_2]_\gamma \Big] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

(In the intermediate steps I have denoted the columns of the matrix as γ-coordinate vectors.) Inverting this matrix, we obtain

$$Q = \left(Q^{-1}\right)^{-1} = [\mathrm{I}]_\gamma^\beta = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

So, as we said above,

$$[\mathrm{T}]_\gamma^\gamma = [\mathrm{I}]_\beta^\gamma \, [\mathrm{T}]_\beta^\beta \, [\mathrm{I}]_\gamma^\beta = Q^{-1} \, [\mathrm{T}]_\beta^\beta \, Q = \begin{bmatrix} -1 & 4 \\ -3 & 7 \end{bmatrix}.$$

Question 7: *Consider the vector space* $W = \{p(x) = ax + bx^2 \,|\, a, b \in \mathbb{R}\}$*. Let* f_1 *and* f_2 *be the linear functionals on* W*, such that* $f_1[p(x)] = p(1)$ *and* $f_2[p(x)] = p(2)$*. Find the basis of* W *to which* $\{f_1, f_2\}$ *is the dual basis.*

Solution: (Based on the framing of the question, you do not need to prove that W is a vector space; however, it is hopefully clear to you that W is a two-dimensional subspace of $\mathrm{P}_2(\mathbb{R})$.) Let's write the desired basis of W as

$$\beta = \left\{a_1x + b_1x^2 \, , \, a_2x + b_2x^2\right\}.$$

By definition of the dual basis $\beta^* = \{f_1, f_2\}$ to β, we must have

$$f_i[a_jx + b_jx^2] = \delta_{ij}$$

for $i = 1, 2$ and $j = 1, 2$. Taking $j = 1$, this leads to a system of equations

$$\begin{cases} f_1[a_1x + b_1x^2] & = 1 \\ f_2[a_1x + b_1x^2] & = 0 \end{cases} \iff \begin{cases} a_1 + b_1 & = 1 \\ 2a_1 + 4b_1 & = 0 \end{cases} \iff \begin{cases} a_1 = 2 \\ b_1 = -1. \end{cases}$$

Similarly, taking $j = 2$ yields

$$\begin{cases} f_1[a_2x + b_2x^2] & = 0 \\ f_2[a_2x + b_2x^2] & = 1 \end{cases} \iff \begin{cases} a_2 + b_2 & = 0 \\ 2a_2 + 4b_2 & = 1 \end{cases} \iff \begin{cases} a_2 = -\frac{1}{2} \\ b_2 = \frac{1}{2}. \end{cases}$$

So the desired basis is

$$\beta = \left\{2x - x^2 \, , \, -\frac{1}{2}x + \frac{1}{2}x^2\right\}.$$

Question 8: *Let V be a finite-dimensional vector space, and $\mathrm{T} : V \to V$ be a linear transformation, represented by matrix $A = [\mathrm{T}]_\beta^\beta$ with respect to some basis β of V.*

Prove that if the kernel of T contains the image of T, then $A^2 = 0$.

Give an example of a transformation satisfying this property.

Solution: Assume that $\mathrm{R(T)} \subseteq \mathrm{N(T)}$. The first thing to notice is that

$$\mathrm{T}^2 := \mathrm{T} \circ \mathrm{T} = \mathrm{T}_0 : V \longrightarrow V,$$

where T_0 denotes the zero transformation on V. Indeed, for any $x \in V$, $\mathrm{T}(x) \in \mathrm{R(T)}$ by definition of the range (or image, using the terminology of the question) of T; by our assumption, this implies $\mathrm{T}(x) \in \mathrm{N(T)}$. So for any $x \in V$, we have

$$\mathrm{T}^2(x) = \mathrm{T}\big(\mathrm{T}(x)\big) = \mathit{0},$$

by definition of the kernel N(T). So $\mathrm{T}^2 = \mathrm{T}_0$ as claimed.

Now we translate the above result to a statement about the matrix representation $A = [\mathrm{T}]_\beta^\beta$. We have

$$A^2 = AA = [\mathrm{T}]_\beta^\beta\,[\mathrm{T}]_\beta^\beta = [\mathrm{T} \circ \mathrm{T}]_\beta^\beta = [\mathrm{T}^2]_\beta^\beta = [\mathrm{T}_0]_\beta^\beta.$$

But the matrix representation of the zero transformation T_0 with respect to *any* basis is the zero matrix, so we have $A^2 = 0$ as desired.

The most obvious example of a transformation satisfying the property $\mathrm{R(T)} \subseteq \mathrm{N(T)}$ is the zero transformation $\mathrm{T} = \mathrm{T}_0 : V \to V$ on any vector space V. But it's instructive to give a nontrivial example of such a T, since more than a few people seemed to think that the condition $\mathrm{R(T)} \subseteq \mathrm{N(T)}$ *implies* that $\mathrm{T} = \mathrm{T}_0$. That is by no means true:

For example, consider $\mathrm{T} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathrm{T}(x, y) = (y, 0)$; you've encountered this transformation on your homework. The kernel of T is the x-axis in $\mathbb{R}^2$:

$$\mathrm{N(T)} = \big\{(x, y) \in \mathbb{R}^2 \,|\, (y, 0) = (0, 0)\big\} = \big\{(x, y) \in \mathbb{R}^2 \,|\, y = 0\big\} = \big\{(x, 0) \in \mathbb{R}^2 \,|\, x \in \mathbb{R}\big\}.$$

But the range of T is also the x-axis:

$$\mathrm{R(T)} = \big\{(y, 0) \in \mathbb{R}^2 \,|\, y \in \mathbb{R}\big\}.$$

So here the kernel of T contains the range of T (in fact, in this case the kernel is equal to the range). But of course T is not the zero transformation on $\mathbb{R}^2$. Taking β to be the standard basis of $\mathbb{R}^2$, you can check that the matrix representation

$$A = [\mathrm{T}]_\beta^\beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

satisfies $A^2 = 0$.