**#1 (30 points): <u>Bernoulli's Law and Manometry</u>**

From hydrostatics, the pressure at section 2 must be

5

$$p_2 = p_{atm} - \rho_w g H$$

Apply Bernoulli's equation between section 1 and section 2:

$$p_1 + \frac{1}{2}\rho_a v_1^2 + \rho_a g z_1 = p_2 + \frac{1}{2}\rho_a v_2^2 + \rho_a g z_2$$

Since $p_1 = p_{atm}$, $p_2 = p_{atm} - \rho_w g H$, and $z_1 = z_2$,

$$\frac{1}{2}\rho_a v_1^2 = -\rho_w g H + \frac{1}{2}\rho_a v_2^2$$

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$$v_2^2 - v_1^2 = 2\frac{\rho_w}{\rho_a} g H$$

Apply continuity between section 1 and section 2:

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$$A_1 v_1 = Q = A_2 v_2$$

$v_1 = Q / A_1$ and $v_2 = Q / A_2$

Lastly,

$$\left(\frac{Q}{A_2}\right)^2 - \left(\frac{Q}{A_1}\right)^2 = Q^2\left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right) = 2\frac{\rho_w}{\rho_a} g H$$

Final correct expression, 10

$$Q = \sqrt{2\frac{\rho_w}{\rho_a} g H \left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right)^{-1}}$$

**#2 (30 points):** **Momentum Analysis of a Control Volume**

a) Since the fluid is inviscid, we can use Bernoulli's equation:

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$$p_1 + \frac{1}{2}\rho v_1^2 = \frac{1}{2}\rho v_e^2 \rightarrow p_1 = \frac{1}{2}\rho(v_e^2 - v_1^2)$$

Based on conservation of mass $\rho A_1 v_1 = \rho A_e v_e \rightarrow v_e = \frac{A_1}{A_e} v_1$ therefore

$$p_1 = \frac{1}{2}\rho v_1^2 (\frac{A_1^2}{A_e^2} - 1)$$

b) Solve for the x- and y-components of the anchoring force to hold the nozzle in terms of variables given. (Define control volume.)

Apply the steady form of the conservation of momentum:

$$\sum F = \int_{CS} \rho \mathbf{v}(\mathbf{v} \cdot \hat{n}) dA$$

Now, in the x-direction, only contribution is from the exit end and $F_x$ is the contact force from the nozzle on the fluid

10

$$F_x = \rho v_e^2 A_e = \rho (A_1 v_1)^2 / A_e$$

In the y-direction, similarly,

$$F_y + p_1 A_1 = \rho v_1(-v_1)A_1 \rightarrow F_y = -p_1 A_1 - \rho v_1^2 A_1$$

$$F_y = -\frac{1}{2}\rho v_1^2 A_1 (\frac{A_1^2}{A_e^2} - 1) - \rho v_1^2 A_1 = -\frac{1}{2}\rho v_1^2 A_1 (\frac{A_1^2}{A_e^2} + 1)$$

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c) Using the moment of momentum equation (steady)

4 bonus

$$M = \int_{CS} (\mathbf{r} \times \mathbf{v}) \rho \mathbf{v} \cdot \hat{n} dA = -\rho h v_e^2 A_e \hat{\mathbf{k}} = -\frac{A_1^2}{A_e} \rho h v_1^2 \hat{\mathbf{k}}$$

where no variation across the nozzle is assumed.

**#3 (30 points): Streaming Flow with a Source**

**a)** Velocity potential

$$\phi = Ur\cos\theta + \frac{m}{2\pi}\ln r$$

Velocity

$$v_r(r,\theta) = \frac{\partial\phi}{\partial r} = U\cos\theta + \frac{m}{2\pi r}$$

$$v_\theta(r,\theta) = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta$$

Full credit for correct expression using stream function as well.

On the stagnation streamline,

$$r\sin\theta = b(\pi-\theta) = \frac{m}{2\pi U}(\pi-\theta)$$

Then $v_r$ along stagnation streamline is

8 $$v_r = U\cos\theta + \frac{m}{2\pi}\frac{2\pi U\sin\theta}{m(\pi-\theta)} = U\left[\cos\theta + \frac{\sin\theta}{\pi-\theta}\right]$$

7 $$v_\theta = -U\sin\theta$$

**b)** Apply Bernoulli's equation:

$$\underbrace{p_0 + \frac{1}{2}\rho U^2}_{\text{Far upstream}} = p(r,\theta) + \frac{1}{2}\rho\left(v_r(r,\theta)^2 + v_\theta(r,\theta)^2\right)$$

Pressure coefficient

$$C_p = \frac{p(r,\theta)-p_0}{\frac{1}{2}\rho U^2} = 1 - \frac{v_r(r,\theta)^2 + v_\theta(r,\theta)^2}{U^2}$$

Apply results from a) for the specific streamline in question:

$$C_p = 1 - \frac{U^2\left[\cos\theta + \frac{\sin\theta}{\pi-\theta}\right]^2 + U^2\sin\theta^2}{U^2}$$

$$= 1 - \left(\left(\cos\theta + \frac{\sin\theta}{\pi-\theta}\right)^2 + \sin\theta^2\right)$$

5 $$= -\frac{\sin\theta}{\pi-\theta}\left(2\cos\theta + \frac{\sin\theta}{\pi-\theta}\right)$$

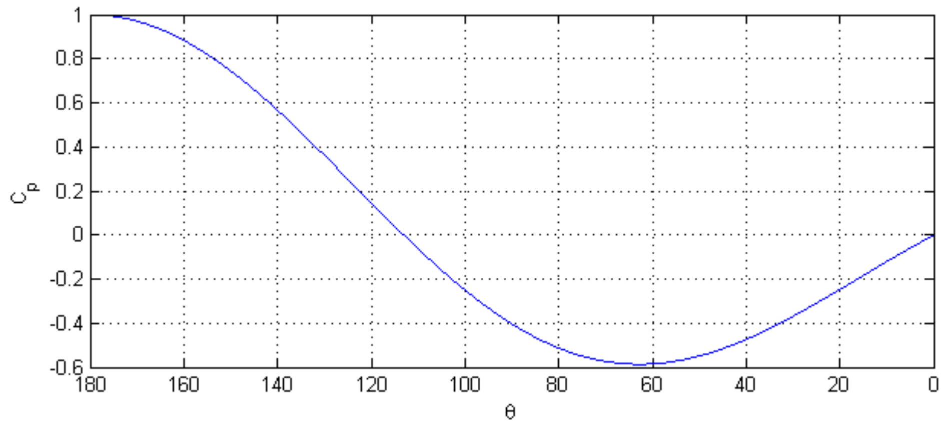
since $\cos^2\theta + \sin^2\theta = 1$ is a well-known trigonometric equation..

**c)** See table of results of evaluation below

10 total for at least 3 evaluations

| $\theta$ (degrees) | *Cp* |
|---|---|
| 60 | -0.5845 |
| 80 | -0.5143 |
| 100 | -0.2525 |
| 120 | 0.1431 |

The following plot is NOT required but done to show the overall behavior of Cp on the stagnation streamline aft of the stagnation point.




**d) Bonus Points**

$\theta = 0°$ corresponds to far downstream of the 'nose.' Pressure is $p_0$ and $C_p = 0$. 2

When $\theta = 90°$, $C_p = -4/\pi^2$ from part **b)**. 1

$\theta = 180°$ corresponds to the stagnation point. $p = p_0 + \frac{1}{2}\rho U^2$ and $C_p = 1$.

Alternatively, we can use the expression form part **b)**. Using L'Hospital's Rule, we have

$$\lim_{\theta\to\pi}\frac{\sin\theta}{\pi-\theta} = \lim_{\theta\to\pi}\frac{\mathrm{d}(\sin\theta)/\mathrm{d}\theta}{\mathrm{d}(\pi-\theta)/\mathrm{d}\theta} = \lim_{\theta\to\pi}\frac{\cos\theta}{-1} = \frac{-1}{-1} = 1$$

$$C_p(\theta=\pi) = \lim_{\theta\to\pi}\left\{-\frac{\sin\theta}{\pi-\theta}\left(2\cos\theta + \frac{\sin\theta}{\pi-\theta}\right)\right\} = -1(-2+1) = 1$$

3

Q#4: 10 points.

Recall from basic calculus: *grad f*(x,y) represents the vector pointing in the direction of the steepest slope on a family of constant-values of *f* curves in the (x,y) plane.

Since $\varphi$ is the velocity potential by definition: $grad\ \varphi = \nabla\varphi \equiv (u,v)$. However, for the stream function $\psi(x,y)$, it is related to the velocities by $u = \dfrac{\partial\psi}{\partial y}, v = -\dfrac{\partial\psi}{\partial x}$ in order to satisfy conservation of mass. Hence the vectors representing the steepest slopes (of rise) of the $\psi(x,y)$ contours are

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$$\nabla\psi = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right) = (-v,u), \quad \text{whereas } \nabla\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right) = (u,v).$$

Thus, $\nabla\varphi \bullet \nabla\psi = \{uv - vu\} = 0$, which implies that *the two families of slope vectors are orthogonal.*

Hence, the constant $\varphi$ and constant $\psi$ curves are orthogonal.

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**Alternative argument, if correctly stated.**

On equi-potential lines or curves: $\mathrm{d}\phi = \dfrac{\partial\phi}{\partial x}\mathrm{d}x + \dfrac{\partial\phi}{\partial y}\mathrm{d}y = u\mathrm{d}x + v\mathrm{d}y = 0$ or the slopes are:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\phi=\text{constant}} = -\frac{u}{v}$$

3

Along streamlines, $\psi$ is constant, and by definition of the stream function,

$$\mathrm{d}\psi = \frac{\partial\psi}{\partial x}\mathrm{d}x + \frac{\partial\psi}{\partial y}\mathrm{d}y = -v\mathrm{d}x + u\mathrm{d}y = 0$$ or the slope of the streamlines are

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\psi=\text{constant}} = \frac{v}{u}$$

3

4

Thus, the slopes of constant potential lines are the INVERSE of the slopes of the streamlines. The constant $\phi$ and constant $\psi$ lines are hence orthogonal.