Midterm 2 – Math 54, April 15, 2015

Please record all work on exam. No calculators.

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1. Always **True** or sometimes **False**?

    1. If A is a square invertible matrix then A and A^{-1} have the same rank. True
    2. The function $y(t) = e^{3t}$ is the only solution to $y''(t) - 6y' + 9y = 0$. False
    3. Each eigenvalue of a square matrix A is also an eigenvalue of A^2. False
    4. There is a t for which the Wronskian of e^t and e^{2t} has determinant zero. False
    5. If A is a diagonalizable square matrix whose eigenvalues are all zero then $e^A = I$. True

1. Invertible means full rank, so A and A^{-1} both have full rank

$\boxed{\text{True}}$

2. $y'' - 6y' + 9y = 0$

$r^2 - 6r + 9 = 0$ $\boxed{\text{False}}$

$(r-3)^2 = 0$

$y_h(t) = c_1 e^{3t} + c_2 t e^{3t}$

3. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ $\boxed{\text{false}}$

$\lambda = 2$ $\qquad \lambda = 4$

4. $\det(W(t)) = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = 2e^{2t} \cdot e^t - e^{2t} \cdot e^t$

$= e^{3t}(2-1) = e^{3t}$

$\boxed{\text{false}}$

$e^{3t} \neq 0$

5. $e^A = P \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix} P^{-1}$

$e^A = P \begin{bmatrix} e^0 & & \\ & \ddots & \\ & & e^0 \end{bmatrix} P^{-1}$ $\boxed{\text{true}}$

$e^A = P \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} P^{-1} = PP^{-1} = I$

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2. Let
$$A = \begin{bmatrix} a & 0 & 1 \\ 0 & b-1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

a) For which pairs of numbers a, b does A have rank 3?

rank = dim Col A

$$\begin{bmatrix} a & 0 & 1 \\ 0 & b-1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} a & 0 & 1 \\ 0 & b-1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} a & 0 & 0 \\ 0 & b-1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 = -R_3 + R_2$ $\qquad$ $R_1 = -R_3 + R_1$

for A to have rank 3, there must be 3 pivot entries in A. Therefore A has rank 3 for all (a,b) such that $a \neq 0$ and $b \neq 1$

b) For which pairs of numbers a, b does A have rank 2?

The row reduced form of A is
$$\begin{bmatrix} a & 0 & 0 \\ 0 & b-1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For A to have rank 2, the must be only 2 pivot entries in A.
Therefore, A has rank 2 for:
(a,b) such that $a = 0$ and $b \neq 1$
(a,b) such that $a \neq 0$ and $b = 1$

$y = e^{-t}$
$y' = -e^{-t}$
$y'' = e^{-t}$

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3. Find all solutions $y(t)$ to the differential equation $y'' - y' - 2y = 4t$.

The auxillary equation is given by

$r^2 - r - 2 = 0$

$\Rightarrow (r-2)(r+1) = 0$

$r_1 = 2 \quad | \quad r_2 = -1$

the general solution for the homogenous equation is given by

$y_h(t) = c_1 e^{2t} + c_2 e^{-t}$, where c_1, c_2 are arbitrary constants

To find the particular solution, use method of undermined coefficients

① $y'' - y' - 2y = 4t$ is in the form $y'' - y' - 2y = Ct^m e^{r_3 t}$

where $m = 1$, $r_3 = 0$, $C = 4$

The guess for the particular solution is

$y_p(t) = (A_1 t + A_0) e^{(0)t} = A_1 t + A_0$

$y_p'(t) = A_1$

$y_p''(t) = 0$

Substituting into ① and matching coefficients,

$0 - A_1 - 2(A_1 t + A_0) = 4t$

$-A_1 - 2A_1 t - 2A_0 = 4t$

$\Rightarrow -A_1 - 2A_0 = 0$

$-2A_1 = 4$

$\Rightarrow A_1 = -2$

$-(-2) - 2A_0 = 0$

$2 - 2A_0 = 0$

$\Rightarrow A_0 = 1$

therefore the perticular solution is given by

$y_p(t) = -2t + 1$

By the superposition priciple all solutions $y(t)$ are given by $y(t) = y_h(t) + y_p(t)$

so $\boxed{y(t) = -2t + 1 + c_1 e^{-t} + c_2 e^{2t}}$

~~$3x^2+2x+4$~~ $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix}4\\2\\3\end{bmatrix} = \begin{bmatrix}8+2+0\\0+4+6\\6\end{bmatrix} = \begin{bmatrix}10\\10\\6\end{bmatrix}$

~~$6x+2+2(3x^2+2x+4)$~~

~~$6x+2$~~ $6x^2+4x+8$

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4. Let P_2 be the vector space of polynomials of degree less than or equal to 2, and let $B = \{1, t, t^2\}$ be the standard ordered basis for P_2. Let $T : P_2 \to P_2$ be the transformation $T(f) = f' + 2f$.

a) Show that T is a linear transformation.

suppose $f \in P_2$ and $g \in P_2$ and c is a constant.

$T(f+g) = (f+g)' + 2(f+g) = f' + g' + 2f + 2g = (f'+2f) + (g'+2g) = T(f)+T(g)$

$= T(f) + T(g)$

$T(cf) = (cf)' + 2(cf) = c(f') + c(2f) = c(f'+2f) = cT(f)$

b) Find the matrix A for T with respect to the basis B.

$A = [[T(1)]_B \; [T(t)]_B \; [T(t^2)]_B]$

$A = [0+2(1) \quad 1+2(t) \quad 2t+2t^2] = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

c) What are the eigenvalues of A?

Since A is a diagonal matrix, eigenvalues are in diagonal

$\lambda = 2$ multiplicity 3

d) What is the kernel of A?

$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\Rightarrow \begin{matrix} 2x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{matrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $\ker A = \{0\}$

e) What are the solutions to $y' + 2y = 0$ that lie in P_2.

$y'+2y=0$ can be written as $L[y] = y'+2y$ where L is a linear operator. $L[y]$ is equivalent to $T[f]$, where $f \in P_2$. The kernel of T was found in part d), which is the same as the solutions to $y'+2y=0$ that lie in P_2.

There fore ~~the~~ solutions that lie in P_2 are $\{y(t)=0\}$