Physics #7B - Fall 2014

Dr. Winoto - Final Exam

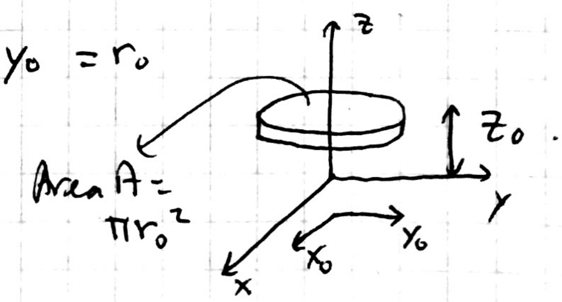
Solution to the Exam 20141218 .

1. Helmholtz coils & anti-Helmholtz coils .

a/. By symmetry (cylindrical symmetry) & Ampere's Law $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$, $B_\phi = 0$ for both (1a) & (1b): Helmholtz & anti-Helmholtz .

For (1a), it is $\boxed{\text{enough}}$ (to get full credit) to show $B_x = B_y = 0$ on the x-y plane , but one can show this more generally in the neighborhood of the origin :

Consider a Gaussian pill box of thickness dz , ~~centered~~ co-axial with the z axis at a distance $z_0 << a$ from the origin ; with radius x_0 or $y_0 = r_0$



$$\oint_S \vec{B}\cdot d\vec{a} = 0 .$$

$$\int_{top} \vec{B}\cdot d\vec{a} + \int_{bottom} \vec{B}\cdot d\vec{a}$$

$$= \int_0^{r_0} B_z(z+dz, r)\, 2\pi r\, dr - \int_0^{r_0} B_z(z, r)\, 2\pi r\, dr \quad ; \quad \vec{B}_r \perp d\vec{a}$$ & does not contribute to flux .

$$= \int_0^{r_0} \frac{\partial B_z}{\partial z}(2\pi r\, dr)\, dz$$

$$\boxed{B_z(z) = B_0 + \text{const.}\, z^4}$$

$$= \int_0^{r_0} (4\,\text{const.}\, z_0^3 \cdot 2\pi r\, dr\, dz$$

$$= \text{const.}\, \pi r_0^2 z_0^3\, dz .$$

$$\int_{\underline{\underline{side}}} \vec{B}\cdot da = B_r \cdot 2\pi r_0\, dz$$

Since $\oint \vec{B}\cdot da = 0 \Rightarrow \int_{top} + \int_{bottom} = -\int_{side}$

$\therefore$

$$B_r \cdot 2\pi r_0\, dz = -\text{const.}\ \pi r_0^2 z_0^3\, dz .$$

$$\boxed{\begin{aligned} |B_r| &\cong \text{const.}\ r_0 z_0^3 . \\ B_z &\cong B_0 + \text{const.}\ z_0^4 \end{aligned}}$$

In the limit: $\frac{r_0}{a} \ll 1$ & $\frac{z_0}{a} \ll 1$:

$$|B_r| = 0 \quad \left(+ \text{ order } (z^4)\right)$$
$$B_z = B_0 \quad \left(+ \text{ order } (z^4)\right)$$

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1b). anti-Helmholtz coils:

$$B_z(z) = \alpha z \hat{z} \quad ; \quad \left.\frac{\partial B_z}{\partial z}\right|_{z=0} = \alpha .$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

Also, just as in 1a). $B_\phi = 0 \Rightarrow \frac{\partial B_x}{\partial y} = 0 = \frac{\partial B_y}{\partial x}$

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

By symmetry: $\frac{\partial B_x}{\partial x} = \frac{\partial B_y}{\partial y}$

$$\therefore \quad 2\frac{\partial B_x}{\partial x} = -\alpha \Rightarrow \frac{\partial B_x}{\partial x} = \frac{\partial B_y}{\partial y} = -\frac{\alpha}{2} .$$

$$\Rightarrow \therefore \quad B_x(x) = -\frac{\alpha}{2} x \hat{x} \quad ; \quad B_y(y) = -\frac{\alpha}{2} y \hat{y} .$$

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2. London eq. & diamagnetism in superconductivity (40 pts):

$$\vec{J} = -\frac{ne^2}{mc}\vec{A}$$

Taking the curl of the Ampere's law and substituting the London equation, we have:

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\times(\mu_0\vec{J})$$

since we have a steady-state time-independent case we can set the displacement current $j_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t} = 0$.

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \vec{\nabla}^2\vec{B} = \vec{\nabla}\times\mu_0\vec{J} = \vec{\nabla}\times\left(-\frac{\mu_0 ne^2}{mc}\vec{A}\right)$$

Since $\vec{\nabla}\cdot\vec{B} = 0$ and $\vec{B} = \vec{\nabla}\times\vec{A}$:

$$\Rightarrow -\vec{\nabla}^2\vec{B} = -\frac{\mu_0 ne^2}{mc}\vec{B}$$

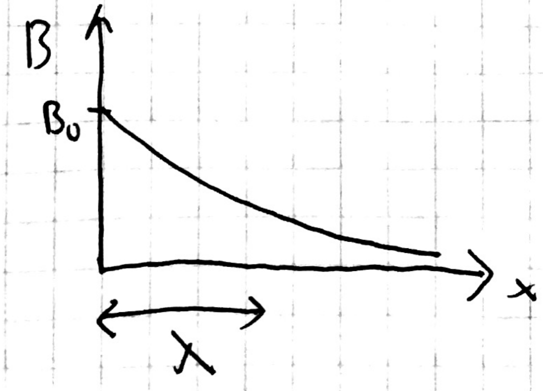
a/. $\Rightarrow \vec{\nabla}^2\vec{B} = \frac{1}{\lambda^2}\vec{B}$ where $\lambda = \left(\frac{mc}{\mu_0 ne^2}\right)^{1/2}$.

b/.

$$\vec{B}(x=0) = B_0\hat{z}, \quad \vec{B}(x\to\infty) = 0.$$

$$\vec{\nabla}^2\vec{B} = \frac{\partial^2\vec{B}}{\partial x^2} = \frac{1}{\lambda^2}\vec{B}. \Leftarrow$$

$$\boxed{\vec{B}(x) = B_0 e^{-x/\lambda}\hat{z}}.$$



(the other solution $e^{+x/\lambda}$ does not satisfy the boundary condition at $x\to\infty$).

$\underline{3.}$ $\vec{E}$ & $\vec{B}$ of a moving electric dipole :

$$V_x = \frac{4}{5}c \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-16/25}} = \frac{5}{3} .$$

$$t' = t = 0 .$$

a/. $x' = d$

$$x = \gamma(x' + vt') = \gamma x' = \gamma d .$$

$$\vec{E}(r = x, \theta = 90^\circ) = \frac{p}{4\pi\epsilon_0} \frac{1}{x^3} (-\hat{z}) .$$

$$= \frac{-p}{4\pi\epsilon_0} \frac{1}{\gamma^3 \cancel{x}^3} \hat{z} .$$

$\to d$.

$$\vec{E}'_z = \gamma E_z$$

$$\vec{E}'(\underset{=d}{x'}, 0, 0) = -\frac{1}{\gamma^2} \frac{p}{4\pi\epsilon_0} \frac{1}{d^3} \hat{z} = \frac{-9}{25} \frac{p}{4\pi\epsilon_0} \frac{1}{d^3} \hat{z}$$

$$\vec{B}'(x', 0, 0) = +\frac{\vec{V}}{c^2} \times \vec{E}' = \frac{36}{125} \frac{p}{4\pi\epsilon_0 c} \frac{1}{d^3} \hat{y} .$$

b/. $z' = d$.

$$z = z' = d .$$

$$\vec{E}(r = z, \theta = 0) = \frac{p}{4\pi\epsilon_0} \frac{2}{z^3} \hat{z} .$$

$$= \frac{p}{4\pi\epsilon_0} \frac{2}{\cancel{z}\, d^3} \hat{z} .$$

$$\vec{E}'(0, 0, z' = d) = \gamma \frac{p}{4\pi\epsilon_0} \frac{2}{d^3} \hat{z} = \frac{10}{3} \frac{p}{4\pi\epsilon_0} \frac{1}{d^3} \hat{z} .$$

$$\vec{B}'(0, 0, z' = d) = +\frac{\vec{V}}{c^2} \times \vec{E}' = -\frac{8}{3} \frac{p}{4\pi\epsilon_0 c} \frac{1}{d^3} \hat{y} .$$

4. Diamagnetism in a classical H :

a/. $F_{centripetal} = F_c$

$$\frac{m v_0^2}{a_0} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{a_0^2}.$$

$$v_0^2 = \frac{1}{m}\frac{1}{4\pi\epsilon_0}\frac{e^2}{a_0}$$

$$v_0 = \sqrt{\frac{1}{m}\frac{1}{4\pi\epsilon_0} e^2/a_0} \approx \frac{1}{137} c .$$

b/. Period $= \frac{2\pi a_0}{v_0} = 1.5 \times 10^{-16}$ sec .

c/. $B_{solenoid} = \vec{B} = \mu_0 n I \hat{z}$

$$\vec{B}(t) = \mu_0 n I(t) \hat{z} \qquad n = 5\times 10^4/m.$$

~~Amperian~~ loop of radius $r = a_0$

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt} \quad ; \quad \vec{E} = E_\phi \text{ by symmetry}.$$

$\dot{B} = \mu_0 n \dot{I}$

$\dot{B} = \mu_0 n \frac{I_0}{T}$.

$$E_\phi \cdot 2\pi r = -\pi r^2 \cdot \dot{B}(t). \quad (r = a_0)$$

$$E = E_\phi = \frac{1}{2} r \dot{B}(t) \quad \text{clockwise (looking from top)}.$$

Force on electron : $F_e = -eE$

$$= \frac{1}{2} e r \dot{B} \quad \text{counterclockwise}.$$

d/. Acceleration :

$$a_\phi = \frac{F_e}{m} = \frac{\frac{1}{2} e r \dot{B}}{m} = \frac{e r \dot{B}}{2m} \quad \# \qquad \boxed{r = a_0}$$

Tangential velocity :

$$\left.\begin{aligned} \int dv &= \int a_\phi \, dt \\ v(t) - v_0 &= a_\phi t \end{aligned}\right\} \Rightarrow v(t) = v_0 + \frac{e r \dot{B}}{2m} t$$

$$= v_0 + \frac{e r B(t)}{2m} \quad \#.$$

4e).

$$\Delta v = v(T) - v_0$$

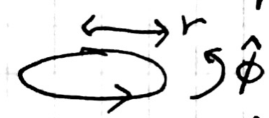
$$= \frac{e a_0 B(T)}{2m} = \frac{e a_0 B(T)}{2m}$$

$$= \frac{e a_0 \mu_0 n I_0}{2m}$$

$$\boxed{\Delta v = 5.8 \text{ m/sec}}$$

5). Displacement current & Induced B-field :

a/. Consider an Amperian loop of radius r deep inside the solenoid :



$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} \qquad \vec{E} = E_\phi \hat{\phi} \text{ by symmetry}$$

$$E_\phi \cdot 2\pi r = -\pi r^2 \frac{dB}{dt} \qquad ; \quad \frac{dB}{dt} = \mu_0 n \omega I_0 \cos \omega t .$$

$$E_\phi = -\frac{1}{2} r \frac{dB}{dt} \hat{\phi}$$

$$\boxed{\Rightarrow E_\phi(r) = -\frac{1}{2} \mu_0 n \omega r I_0 \cos \omega t .}$$

b/. $$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t} = +\frac{1}{2} \epsilon_0 \mu_0 n \omega^2 r I_0 \sin \omega t \ \hat{\phi}$$

$$= \frac{1}{2} \frac{\omega^2}{c^2} r n I_0 \sin \omega t \qquad \text{since } c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} .$$

5c).

Induced B-field:

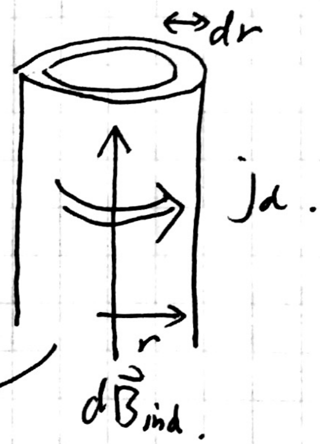
The modified Ampere's Law:

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \int_S (\vec{J} + \vec{J}_d)\cdot d\vec{a}.$$

Let's consider the contribution from J_d only:

$$\oint_C \vec{B}_{ind}\cdot d\vec{l} = \mu_0 \int_S \vec{J}_d \cdot d\vec{a}.$$

$\vec{J}_d$ is cylindrical current / circularly symmetric and linearly proportional to $|\vec{r}|$. We can calculate the B-field $\vec{B}_{ind}$ due to $\vec{J}_d$ by calculating the B-field due to solenoid of J_d: radius r and thickness dr, and integrating over dr:



surface current: $\vec{\mathcal{J}}_d(r) = \vec{J}_d(r)\cdot dr$

$$dB_{ind} = \mu_0 \vec{\mathcal{J}}_d(r)\,\hat{z}$$
$$= \mu_0 J_d(r)\,dr\,\hat{z}$$

at $r=0$
$$B_{ind} = \int_0^a dB_{ind} = \frac{1}{2}\mu_0 n I_0 \frac{\omega^2}{c^2}\sin\omega t \int_0^a r\,dr.$$

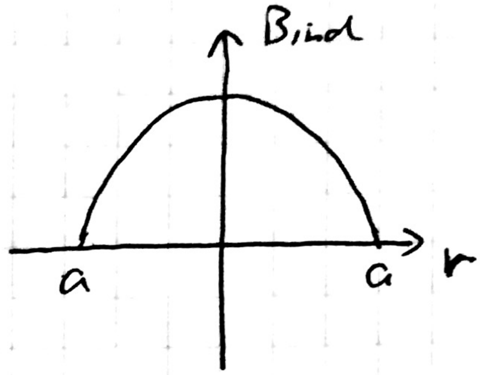
$$B_{ind}(0) = \frac{1}{4}\left(\frac{\omega^2}{c^2}a^2\right)\mu_0 n I_0 \sin\omega t.$$

Since dB inside solenoid $= \mu_0 \mathcal{J}$
$= 0$ outside of solenoid.

then: $$B_{ind}(r) = \int_r^a dB_{ind} = \frac{1}{2}\mu_0 n I_0 \frac{\omega^2}{c^2}\sin\omega t \int_r^a r\,dr.$$

5c) con't :

$$B_{ind}(r) = \frac{1}{4}\frac{\omega^2}{c^2}(a^2 - r^2)\mu_0 n I_0 \sin\omega t .$$



$$B_{tot} = B_{solenoid} + B_{ind}(r)$$

$$B_{tot}(r) = \mu_0 n I_0 \sin\omega t \left[1 + \frac{1}{4}\frac{\omega^2}{c^2}(a^2 - r^2)\right]$$

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d). Extra credit :

Vector potential :

$$\oint \vec{A}\cdot d\vec{l} = \int_S \vec{B}\cdot da$$

$$A_\phi \cdot 2\pi r = \underbrace{\int_{S(r)} B\cdot da}_{\Phi(r)}$$ for choice of gauge $\vec{\nabla}\cdot\vec{A} = 0$.

$$\Phi(r) = \int_0^r B_{tot}(r)\cdot 2\pi r\, dr$$

$$= 2\pi\mu_0 n I \sin\omega t \int_0^r \left[1 + \frac{1}{4}\frac{\omega^2}{c^2}(a^2 - r^2)\right] r\, dr$$

$$= 2\pi\mu_0 n I \sin\omega t \left[\frac{1}{2}\left(1 + \frac{1}{4}\frac{\omega^2}{c^2}a^2\right)r^2 - \frac{1}{16}\frac{\omega^2}{c^2}r^4\right].$$

$$A_\phi(r) = \mu_0 n I \sin\omega t \left[\frac{1}{2}\left(1 + \frac{1}{4}\frac{\omega^2}{c^2}a^2\right)r - \frac{1}{16}\frac{\omega^2}{c^2}r^3\right].$$

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